PHYSICAL REVIEW B **79**, 125110 (2009)

Nonequilibrium critical behavior for electron tunneling through quantum dots in an Aharonov-Bohm circuit

Eran Sela and Ian Affleck

Department of Physics and Astronomy, University of British Columbia, Vancouver, British Columbia, Canada V6T 1Z1

(Received 29 December 2008; published 13 March 2009)

Double quantum dots can provide an experimental realization of the two-impurity Kondo model which exhibits a non-Fermi-liquid quantum critical point (QCP) at a special value of its parameters. We generalize our recent study of double quantum dots in series [E. Sela and I. Affleck, Phys. Rev. Lett. **102**, 047201 (2009)] to a parallel configuration with an Aharonov-Bohm flux. We present an exact universal result for the finite temperature and finite voltage conductance $G[V,T]$ along the crossover from the QCP to the low energy Fermi-liquid phase. Compared to the series configuration, here generically $G[V,T]\neq G[-V,T]$, leading to current rectification.

DOI: 10.1103/PhysRevB.79.125110 PACS number(s): 75.20.Hr, 71.10.Hf, 75.75.+a, 73.21.La

I. INTRODUCTION

It is now well established that quantum dots (QDs) behave as Kondo impurities at low temperatures.[1,2] Whereas many theoretical tools are available to address linear transport, the nonequilibrium regime is far less studied, although it is typically addressed in experiment.[3] A solution of nonequilibrium transport through a one-channel Kondo impurity was achieved in Ref. 4; however exact results were obtained only for a specific point in the parameter space (Toulouse limit). Another important development in this direction was the application of the Bethe ansatz and finding of many body scattering states.[5,6] Recently we found exact results for nonlinear transport close to a quantum critical point (QCP) in a double dot in series realizing the two-impurity Kondo model (2IKM), and showing non-Fermi liquid (NFL) behavior.[7]

The 2IKM consists of two-impurity spins coupled to two channels of conduction electrons and, at the same time, interacting with each other through an exchange interaction K. Jones *et al.*[8] observed that a QCP at $K=K_c$ separates a "local singlet" from a Kondo-screened phase, where K_c is of the order of the Kondo temperature T_K. The exact critical behavior was found using conformal field theory[9,10] (CFT) and Abelian bosonization[11] methods. Implications of the 2IKM for transport through double QDs were studied in Refs. 12–19.

The presence of a sharp quantum phase transition in the 2IKM became questionable soon after its discovery; in the mean field study in Ref. 20 it was pointed out that the true QCP is restricted to the case of a special particle hole (PH) symmetry assumed in Ref. 8. This was confirmed by numerical renormalization group calculations.[21] PH symmetry breaking was later associated with two relevant potential scattering (PS) perturbations.[10,17] Thus, in real systems the critical behavior for $K=K_c$ can be observed only above a certain crossover energy scale, denoted here as T_{LR}^{*}. In order to obtain reliable predictions for QDs it is crucial to include the extra relevant perturbations associated with potential scattering in a real calculation. We achieved this task for a double QD (Ref. 7) using the method developed by Gan.[11] The finding of exact crossover results including PH symmetry breaking remains an open problem for the alternative proposed realization of the 2IKM by Zaránd *et al.*[17] Compared to their QD system involving at least three leads, our system has only two leads making the nonequilibrium behavior more tractable.

In this paper we generalize our previous results to a generic configuration ranging from series to parallel QD attached to two leads [see Fig. 1]. In this generic configuration transport from left to right occurs via different interfering paths. A particular feature of our results distinguishes the generic case from the series case: in the generic case the finite voltage conductance $G[V]$ has the property $G[V]\neq G[-V]$, leading to current rectification, similar to a diode. This effect results from interactions and is absent in a noninteracting Landauer description.[22] An additional aim of this paper is to provide important details on the calculation for the general series or parallel cases.

The outline of the paper is as follows. In Sec. II the double QD system is presented and mapped to the 2IKM. In Sec. III the conductance is calculated at the QCP using CFT methods, neglecting the effect of potential scattering. In Sec. IV we consider deviations from the QCP due to variations in K from K_c and calculate the finite temperature crossover for the linear conductance using a mapping of the PH symmetric 2IKM to the Ising model with a boundary magnetic field. We also apply this mapping for the QD system proposed by Zaránd *et al.*[17] as a realization of the 2IKM. In Sec. V potential scattering is incorporated in the Hamiltonian close to the QCP and in the crossover formula for the linear conduc-

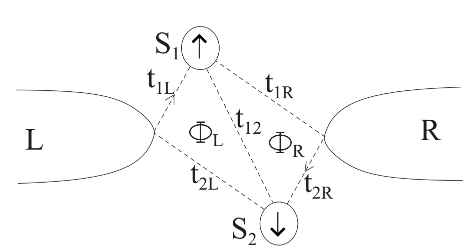

FIG. 1. Schematic description of the physical system, consisting of two leads (L and R) and two quantum dots with effective spin 1/2. We use the convention that for finite flux $t_{1L}\rightarrow t_{1L}e^{i\Phi_L}$ and $t_{2R}\rightarrow t_{2R}e^{i\Phi_R}$, where the t_{iL} are defined as tunneling amplitudes from lead $(L,R)\rightarrow$dot (i).

tance. In Sec. VI the full nonequilibrium problem at finite voltage and temperature in the vicinity of the critical point is addressed. Section VII contains conclusions. We relegate details on the calculation of the nonlinear conductance using Keldysh Green's functions (GFs) to the Appendix.

II. MODEL

The physical system under consideration is shown schematically in Fig. 1. It consists of left (L) and right (R) leads tunnel coupled to two quantum dots 1, 2, with tunneling amplitudes $t_{iL/R}$, ($i=1,2$). We assume that both dots are in the Kondo regime, with gate voltages adjusted to give an odd number of electrons, and the $t_{iL/R}$ are sufficiently weak compared to the charging energy, U, so that charge fluctuations can be ignored. We write the effective spin-1/2 moments as $\vec{S}_1$ and $\vec{S}_2$. We will be primarily interested in the case where $t_{1L},t_{2R} \gg t_{1R},t_{2L}$ so that the left lead is primarily coupled to dot 1 and the right lead to dot 2 since only in this case will the QCP occur. Note that in the extreme case where $t_{1R}=t_{2L}=0$, this reduces to the series configuration analyzed, for example, in Ref. 7. The fluxes Φ_L and Φ_R are introduced in the triangular plaquettes as shown.

In the standard fashion,[23] the conduction-electron channels that couple to the impurity are reduced to one-dimensional left moving Dirac fields $\psi_{i\alpha}(x)$, where $i=L,R$ and $\alpha=\uparrow,\downarrow$ are the lead and spin indices, respectively. We assume that a *single mode* in each lead couples to both impurities. Here we have linearized the conduction-electron dispersion around the Fermi level: $\epsilon_k=\hbar v_F k$, where ϵ_k and k are measured relative to the Fermi level and Fermi wave number, respectively. x is a fictitious position variable conjugate to k. We set $\hbar=v_F=1$.

We discuss the different terms which will appear in the model Hamiltonian [Eq. (2.2)]. An exchange interaction

$$K_{12} \sim \frac{t_{12}^2}{U} \tag{2.1}$$

between the impurity spins is generated by the interimpurity tunneling t_{12}. The impurity spins are also Kondo coupled to the conduction-electron spin density at the origin,

$$\vec{s}_j^{\,i} = \psi^{\dagger i\alpha}\frac{\vec{\sigma}_\alpha^\beta}{2}\psi_{j\beta},$$

$$i,j = L,R = 1,2 \quad \text{(repeated spin indices summed)}.$$

In addition there are PS terms $\propto \psi^{\dagger i\alpha}\psi_{j\alpha}$ (repeated spin indices summed). The system is driven out of equilibrium by a source drain voltage V. Thus, the Hamiltonian H is

$$H = H_0 + H_V + K_{12}\vec{S}_1\cdot\vec{S}_2 + H_K + H_{\rm PS} + H',$$

$$H_0 = \int_{-\infty}^{\infty} dx\,\psi^{\dagger j\alpha} i\partial_x \psi_{j\alpha},$$

$$H_V = \frac{eV}{2}\int_{-\infty}^{\infty} dx\,\psi^{\dagger i\alpha}(\tau^z)_i^j\psi_{j\alpha},$$

$$H_K = \sum_{\ell=1,2}(J^{(\ell)})_i^j\vec{s}_j^{\,i}\cdot\vec{S}_\ell,$$

$$H_{\rm PS} = \psi^{\dagger i\alpha}V_i^j\psi_{j\alpha}, \tag{2.2}$$

with repeated lead and spin indices summed. To be complete one has to add the terms

$$H' = V_i''^j\vec{s}_j^{\,i}\cdot(\vec{S}_1\times\vec{S}_2) + \psi^{\dagger i\alpha}V_i'^j\psi_{j\alpha}(\vec{S}_1\cdot\vec{S}_2).$$

However, close to the QCP the first (second) term of H' has a similar effect as H_K ($H_{\rm PS}$). Therefore, up to a correction to the actual coupling constants, energy scales, and to the critical value of different parameters at the QCP, all of which we are not able to determine exactly, it is legitimate to neglect H'.

The Kondo interaction induces, via the Ruderman-Kittel-Kasuya-Yosida (RKKY) mechanism, an additional contribution to the interimpurity exchange, $K=K_{12}+K_{\rm RKKY}$, where

$$K_{\rm RKKY} = 2\left\langle S_1=\downarrow,S_2=\uparrow\left|H_K\frac{1}{-H_0}H_K\right|S_1=\uparrow,S_2=\downarrow\right\rangle$$

$$= 4\sum_{k_1>0,k_2<0}\frac{1}{-(\epsilon_{k_1}-\epsilon_{k_2})}{\rm tr}\{J^{(1)}J^{(2)}\}.$$

With the parametrization of $J^{(\ell)}$ given in Eq. (2.6), ${\rm tr}\{J^{(1)}J^{(2)}\}=4J^2\sin^2(2\theta)\cos^2\frac{\Phi}{2}$. Using $\sum_k=\nu\int d\epsilon$, where ν is the density of states in the leads, and restricting the bandwidth to $|\epsilon_k|<U$, beyond which the effective spin description breaks down, one obtains a ferromagnetic contribution,

$$K_{\rm RKKY} \sim -(\nu J)^2 U\sin^2(2\theta)\cos^2\frac{\Phi}{2}. \tag{2.3}$$

We estimate the potential scattering amplitudes by

$$V_i^i \sim \frac{t_{1i}^2+t_{2i}^2}{U} \quad (i=L,R),$$

$$V_L^R \sim \frac{t_{1L}t_{1R}e^{i\Phi_L}+t_{2L}t_{2R}e^{-i\Phi_R}}{U} + c'\frac{t_{1L}t_{12}t_{2R}e^{i(\Phi_L-\Phi_R)}+t_{2L}t_{12}t_{1R}}{U^2}, \tag{2.4}$$

where c' is a constant factor of order 1. Until Sec. VI A we will assume the parity symmetry

$$S_1 \leftrightarrow S_2, \quad L\leftrightarrow R. \tag{2.5}$$

However our results are not restricted to this case, as will be discussed in Sec. VI A. Parity implies $t_{1L}=t_{2R}\equiv t_1$; $t_{2L}=t_{1R}\equiv t_2$. For finite flux $\Phi=\Phi_L+\Phi_R$ the parity symmetry is preserved for $\Phi_L=\Phi_R$. Calculating the Kondo couplings to second order in the tunneling amplitudes, under this symmetry, gives $\{J^{(1)},J^{(2)}\}\propto\{\frac{vv^\dagger}{U},\frac{\tau^x vv^\dagger\tau^x}{U}\}$, where $v=\binom{t_1e^{i\Phi/2}}{t_2}$. This leads us to parametrize the Hermitian exchange matrices by

$$J^{(1)}=\hat{J},\quad J^{(2)}=\tau^x\hat{J}\tau^x,\quad \hat{J}=J\{1+\cos(2\theta)\tau^z+\sin(2\theta)\times[\cos(\Phi/2)\tau^x-\sin(\Phi/2)\tau^y]\}, \tag{2.6}$$

where

$$\theta = |\arctan(t_2/t_1)|, \quad J \sim \frac{t_1^2+t_2^2}{U}. \tag{2.7}$$

Parity symmetry for the PS amplitudes implies $V_L^L = V_R^R$, $V_R^L = V_L^R$ ($\mathrm{Im}\ V_L^R = 0$). We can estimate

$$V_L^L \sim \frac{t_1^2+t_2^2}{U}, \quad V_R^L \sim \frac{t_1 t_2}{U}\cos\frac{\Phi}{2} + c'\frac{(t_1^2+t_2^2)t_{12}}{U^2}. \tag{2.8}$$

It is convenient to define even and odd channels $\psi_{e,o} = \frac{\psi_L \pm \psi_R}{\sqrt{2}}$ in terms of which the parity transformation reads $\psi_e \to \psi_e$, $\psi_o \to -\psi_o$. The most general form of $H_K + H_{\mathrm{PS}}$ consistent with parity is

$$H_K = J_e \psi^{\dagger e\alpha} \frac{\vec{\sigma}_\alpha^\beta}{2} \psi_{e\beta} \cdot (\vec{S}_1 + \vec{S}_2) + J_0 \psi^{\dagger o\alpha} \frac{\vec{\sigma}_\alpha^\beta}{2} \psi_{o\beta} \cdot (\vec{S}_1 + \vec{S}_2) + \left[J_m \psi^{\dagger e\alpha} \frac{\vec{\sigma}_\alpha^\beta}{2} \psi_{o\beta} + \mathrm{H.c.} \right] \cdot (\vec{S}_1 - \vec{S}_2),$$

$$H_{\mathrm{PS}} = V_e \psi^{\dagger e\alpha} \psi_{e\alpha} + V_o \psi^{\dagger o\alpha} \psi_{o\alpha}. \tag{2.9}$$

Indeed using Eq. (2.6) we obtain $H_K + H_{\mathrm{PS}}$ in this form with

$$J_{e,o} = \frac{\hat{J}_L^L + \hat{J}_R^R \pm (\hat{J}_L^R + \hat{J}_R^L)}{2} = J[1 \pm \sin(2\theta)\cos(\Phi/2)],$$

$$J_m = \frac{\hat{J}_L^L - \hat{J}_R^R + \hat{J}_L^R - \hat{J}_R^L}{2} = J[\cos(2\theta) - i\sin(2\theta)\sin(\Phi/2)] = |J_m| e^{i\phi_m},$$

$$V_{e,o} = \frac{V_L^L + V_R^R \pm (V_L^R + V_R^L)}{2}, \tag{2.10}$$

where

$$\phi_m = -\arctan[\tan(2\theta)\sin(\Phi/2)]. \tag{2.11}$$

For finite flux J_m has an imaginary part. To recover real coupling constants in H_K we remove this phase by a redefinition of the fields,

$$\psi_e \to \psi_e' = e^{-i\phi_m/2}\psi_e,$$

$$\psi_o \to \psi_o' = e^{i\phi_m/2}\psi_o. \tag{2.12}$$

In the $\psi_{e,o}'$ basis, H_K has real coupling constants, J_e, J_o, and $|J_m|$, and it corresponds to the notation in Ref. 10. It is convenient to define $\psi_1' = \frac{\psi_e' + \psi_o'}{\sqrt{2}}$, $\psi_2' = \frac{\psi_e' - \psi_o'}{\sqrt{2}}$. Equivalently, the fields ψ_j' ($j=1,2$) are related to the L-R basis by the rotation $\psi_i = (\mathcal{M} e^{i\tau^z \phi_m/2} \mathcal{M})_i^j \psi_j'$, where $\mathcal{M} = \frac{\tau^z + \tau^x}{\sqrt{2}}$.

As will be discussed in Sec. VI A, observability of the QCP in this system is restricted to the regime $t_1 \gg t_2$ or equivalently small θ [see Eq. (2.7)]. In this limit the two-impurity Kondo physics is especially transparent since each QD is coupled essentially to one lead.[7] K can be tuned by means of t_{12}. For $K \gg K_c$ the impurities are locked into a singlet, while for $K=0$ each impurity is Kondo screened by the nearby lead. In the case of exact PH symmetry, occurring for $V_i^j = 0$, those points in the K-parameter space are separated by a QCP at a critical value $K = K_c \sim T_K$.[8]

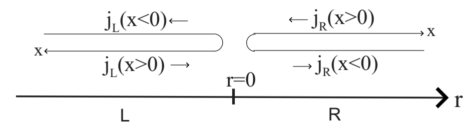

FIG. 2. Illustration of the physical coordinate r running from left to right leads and the fictitious coordinate x labeling position of the chiral fields $\psi_{i\alpha}(x)$ in lead $i=L,R$.

III. CONDUCTANCE AT THE QUANTUM CRITICAL POINT

In Ref. 7 the conductance of a series double QD was calculated using the tunneling current operator. In this paper until Sec. VI we use the Kubo linear conductance formula written in terms of *bulk* current correlation function. The reason for taking this different approach here is that it relates the conductance to correlation functions in certain field theories, which can be addressed using boundary conformal field theory or integrability methods. This allows us to express the conductance of double QDs described by the 2IKM in terms of correlation functions in the boundary Ising field theory.

The linear conductance can be calculated from the Kubo formula,

$$G = \lim_{L\to\infty} \lim_{\omega\to 0} \frac{e^2}{\hbar\omega(2L)^2} \int_{-L}^{L} dr \int_{-L}^{L} dr' \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \times \langle J(r,\tau) J(r',0) \rangle.$$

Here r is the physical coordinate (see Fig. 2). It should be distinguished from the fictitious coordinate x labeling the chiral fermions $\psi_{i\alpha}(x)$. We define the chiral current densities in each lead $j_L(x) = \psi^{\dagger L\alpha}(x)\psi_{L\alpha}(x)$, $j_R(x) = \psi^{\dagger R\alpha}(x)\psi_{R\alpha}(x)$. The bulk current operator $J(r)$ can be written as

$$J(r) = -\begin{cases} j_R(r) - j_R(-r), & r > 0 \\ j_L(r) - j_L(-r), & r < 0. \end{cases}$$

It is useful to define the odd current $j_o(x) = j_L(x) - j_R(x)$ since $\int_{-L}^{L} dr J(r) = \int_{-L}^{L} dx\, j_o(x)\mathrm{sgn}(x)$. The conductance is given in terms of the odd current correlator,

$$G = \lim_{L\to\infty} \lim_{\omega\to 0} \frac{e^2}{\hbar\omega(2L)^2} \int_{-L}^{L} dx \int_{-L}^{L} dy \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \times \langle j_o(x,\tau) j_o(y,0) \rangle \mathrm{sgn}(xy). \tag{3.1}$$

The odd current $j_o(x) = \psi^{\dagger j\alpha}(\tau^z)_j^i \psi_{j\alpha}$ corresponds to the z component of the flavor current of the fermions in the L-R basis. We define the flavor current in terms of the fermions $\psi_{j\alpha}'$ after the rotation [Eq. (2.12)],

$$\vec{j}^f = \psi'^{\dagger i\alpha} \frac{\vec{\tau}_i^j}{2} \psi_{j\alpha}' \quad \text{(repeated indices summed)}. \tag{3.2}$$

The transformation [Eq. (2.12)] amounts to a rotation in the flavor sector,

$$j_o = 2[\cos\phi_m (j^f)^z - \sin\phi_m (j^f)^y]. \tag{3.3}$$

Consider the weak-coupling limit $J\to 0$. It corresponds to a trivial boundary condition (BC) $\psi_L(x=0^+)=\psi_L(x=0^-)$, $\psi_R(x=0^+)=\psi_R(x=0^-)$, describing free fermions with full reflection at the boundary. Also this BC makes apparent the continuity of the chiral fields ψ_L, ψ_R at $x=0$. Accordingly, the odd current correlator is given by

$$\langle j_o(x,\tau) j_o(y,0)\rangle_{J=0} = -\frac{1}{\pi^2}\frac{1}{[\tau+i(x-y)]^2}.$$

To calculate the odd current correlator at the nontrivial fixed point, we apply CFT methods and the Bose-Ising representation used in Ref. 10. In this representation the four fermions $\psi'_{i\alpha}$ are represented using a coset construction in terms of three Wess-Zumino-Witten (WZW) nonlinear σ models, $SU(2)_1^{\text{charge1}}\times SU(2)_1^{\text{charge2}}\times SU(2)_2^{\text{spin}}$, together with a $\mathcal{Z}_2$ Ising model. The currents of the two $SU(2)_1$ σ models are associated with the charge of each species $\psi'_{1\alpha}$ and $\psi'_{2\alpha}$. The current of the $SU(2)_2$ model is associated with the total spin.

Following Ref. 10 one may write down representations of the various operators in the free fermion theory as product of charge (or isospin) bosons, the total spin boson, and the Ising field. The $k=2$ WZW model has primary fields of spin $j=0$ (identity operator $\mathbf{1}$), $j=1/2$ (fundamental field g_α), and $j=1$ (denoted $\vec{\phi}$). The $k=1$ WZW model only has the identity operator and the $j=1/2$ primary, h_A. Their scaling dimension is given by $\Delta=\frac{j(j+1)}{2+k}$. The Ising model has three primary fields: the identity operator $\mathbf{1}$ ($\Delta=0$), the Ising order parameter σ ($\Delta=1/16$), and the energy operator ϵ ($\Delta=1/2$). For example, the fermion field is written in this representation as

$$\psi'_{i\alpha}\propto (h_i)_1 g_\alpha \sigma. \tag{3.4}$$

The three factors have dimensions which add correctly to 1/2. The representation of other operators can be determined using the operator product expansion (OPE). For the Ising model the OPE gives

$$\sigma\times\sigma\to 1+\epsilon,\quad \sigma\times\epsilon\to\sigma,\quad \epsilon\times\epsilon\to\mathbf{1}.$$

This OPE is equivalent to that of the $k=2$ WZW model with the identifications $\sigma\leftrightarrow g$ and $\epsilon\leftrightarrow\vec{\phi}$.

Using the OPE, symmetry considerations, and consistency of scaling dimensions, we shall determine the representation of the odd current j_o. The latter is related in Eq. (3.3) to the flavor current operators $(j^f)^z$ and $(j^f)^y=\frac{(j^f)^+-(j^f)^-}{2i}$. First consider $(j^f)^z=\frac{1}{2}(\psi'^{\dagger 1\alpha}\psi'_{1\alpha}-\psi'^{\dagger 2\alpha}\psi'_{2\alpha})$. This is just the charge difference between flavors, represented by $I_1^z-I_2^z$, where $\vec{I}_i$ is the $SU(2)_1^{\text{charge}i}$ current ($i=1,2$). For the operator $(j^f)^+=\psi'^{\dagger 1\alpha}\psi'_{2\alpha}$, we use Eq. (3.4),

$$\psi'^{\dagger 1\alpha}(x)\psi'_{2\alpha}(x)\propto \lim_{x'\to x} g^{\alpha\dagger}(x')g_\alpha(x)(h_1)^{1\dagger}(x') \times (h_2)_1(x)\sigma(x')\sigma(x).$$

Consider the OPE of the fundamental field $g\times g=\mathbf{1}+\vec{\phi}$. The operator under consideration is a spin singlet, ruling out $\vec{\phi}$ in the OPE. To account for the consistency of scaling dimensions we must have $\sigma\times\sigma\to\epsilon$. Hence $\psi'^{\dagger 1\alpha}\psi'_{2\alpha}\propto (h_1)^{1\dagger}(h_2)_1\epsilon$. The Bose-Ising representation of the flavor current is summarized in the first column of Table I. In the second column we consider an alternative SO(8) representation introduced in Sec. VI.

TABLE I. Bose-Ising versus SO(8) Majorana representation of the flavor current.

	$SU(2)_1\times SU(2)_1\times SU(2)_2\times\mathcal{Z}_2$	SO(8)
$(j^f)^z$	$I_1^z-I_2^z$	$\psi_f^\dagger\psi_f$
$(j^f)^+$	$(h_1)^{1\dagger}(h_2)_1\epsilon$	$\psi_f^\dagger\chi_2^X$

The main result in Ref. 10 is that the nontrivial BC of the 2IKM at $K=K_c$ corresponds to a change in the boundary condition occurring only in the Ising sector of the theory: the nontrivial BC of the electrons corresponds to the free BC on the Ising chain, whereas the trivial BC for the electrons corresponds to the Ising model with a fixed-boundary spin.[24] We shall refer sometimes to the BC of the full system at the nontrivial fixed point by "free" and at the trivial free fermion fixed point by "fixed."

The remaining sectors of the theory other than the Ising model remain unaffected. Correlation functions of factors belonging to sectors other than the Ising model have the form dictated by conformal invariance, $\langle\mathcal{O}_\Delta(x,\tau)\mathcal{O}_\Delta(y,0)\rangle=\frac{1}{[\tau+i(x-y)]^{2\Delta}}$, where Δ is the scaling dimension of $\mathcal{O}$. This form remains valid both at the trivial and nontrivial fixed points. On the other hand correlation functions of fields from the Ising sector do depend on BC. There is a general formula for correlation function of primary operators for a BC obtained by fusion with a primary a,[25,26]

$$\langle\mathcal{O}_\Delta(x,\tau)\mathcal{O}_\Delta(y,0)\rangle=\frac{1}{[\tau+i(x-y)]^{2\Delta}}\begin{cases}1, & xy>0\\ \dfrac{S_a^\Delta/S_0^\Delta}{S_a^0/S_0^0}, & xy<0.\end{cases} \tag{3.5}$$

Here S_j^a are elements of the modular S matrix. For the Ising model this is given by

$$S=\begin{pmatrix}1/2 & 1/2 & 1/\sqrt{2}\\ 1/2 & 1/2 & -1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2} & 0\end{pmatrix},$$

where the first, second, and third rows and columns are labeled by the fields with scaling dimension 0, 1/2, and 1/16, respectively. The change in BC in the 2IKM from trivial to nontrivial fixed points corresponds to fusion with the spin operator in the Ising sector.[10] Setting $\Delta=1/2, a=1/16$ we have $\frac{S_a^\Delta/S_0^\Delta}{S_a^0/S_0^0}=-1$. Hence Eq. (3.5) gives

$$\langle\epsilon(x,\tau)\epsilon(y,0)\rangle_{\text{free}}=\langle\epsilon(x,\tau)\epsilon(y,0)\rangle_{\text{fixed}}\cdot\text{sgn}(xy), \tag{3.6}$$

where up to a normalization factor $\langle\epsilon(x,\tau)\epsilon(y,0)\rangle_{\text{fixed}}\propto\frac{1}{\tau+i(x-y)}$. We may interpret this as a phase shift of $\pi/2$ that the energy operator $\epsilon(x)$ undergoes at $x=0$. We proceed to

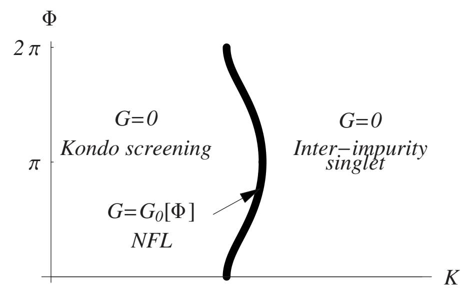

FIG. 3. Schematic phase diagram as a function of interimpurity interaction K (controlled by t_{12}) and flux Φ for fixed asymmetry (we assume a generic situation $t_1 \neq t_2$). Here PH symmetry is assumed. The conductance is finite only at the NFL curve defined by $K=K_c[\Phi]$ except for $\Phi=0,2\pi$.

evaluate the odd current correlation function. Since the crossed terms $\langle (j^f)^z (j^f)^y \rangle$ vanish, we obtain

$$\langle j_o(x,\tau) j_o(y,0)\rangle_{\text{free}} = \langle j_o(x,\tau) j_o(y,0)\rangle_{J=0} \times \begin{cases} 1, & xy > 0 \\ \cos(2\phi_m), & xy < 0. \end{cases} \tag{3.7}$$

One can use the Kubo formula [Eq. (3.1)] to calculate the conductance. However a calculation is unnecessary: curiously, one obtains exactly the same result for the odd current correlation function [Eq. (3.7)], assuming free fermions with partially transmitting BC,

$$\psi_L(0^+) = \cos(\phi_m)\psi_L(0^-) + i\,\sin(\phi_m)\psi_R(0^-),$$

$$\psi_R(0^+) = i\,\sin(\phi_m)\psi_L(0^-) + \cos(\phi_m)\psi_R(0^-). \tag{3.8}$$

This BC corresponds to transmission probability $\sin^2\phi_m$ per spin. From Landauer formula[22] the linear conductance at the nontrivial fixed point is

$$G_0 = \frac{2e^2}{h}\sin^2\phi_m, \tag{3.9}$$

where ϕ_m is given in Eq. (2.11). This gives $G_0=0$ for $\phi_m=0$, in particular for $\Phi=0$ and for the case of a series QD (in this section $V_L^R=0$).[7] Also in the experimentally relevant regime $t_1 \gg t_2$ (see Sec. VI A), corresponding to small θ [see Eq. (2.7)], $G_0 \ll 2e^2/h$. [Note however, that the actual BC at the nontrivial fixed point written in terms of the true fermions is very different from Eq. (3.8). This is apparent from the vanishing of the one-particle S-matrix.[10] The auxiliary fermions satisfying linear boundary condition emerge in the SO(8) representation that we shall use in Sec. VI.]

A. $T=0$ phase diagram

Having found the conductance at the QCP at $K=K_c$, we shall consider the surrounding FL fixed points and draw a phase diagram. Here and until Sec. V we consider the PH symmetric model. In this model charge transfer between the leads leading to finite current is allowed by the exchange interaction in H_K. The main role of $\theta=\arctan(t_2/t_1)$ and flux Φ is to modify the crossover scales T_K and K_c. We plot in Fig. 3 the phase diagram at fixed θ as a function of K and flux. The NFL state occurs along the curve $K=K_c[\Phi]$, where $K_c \sim T_K[J_e,J_o,J_m]$ and J_e,J_o,J_m depend on flux through Eq. (2.10). This curve is characterized by a finite conductance $G=G_0$ at $\Phi \neq 0,2\pi$. It separates the $K>K_c$ local singlet phase from the $K<K_c$ Kondo-screened phase.

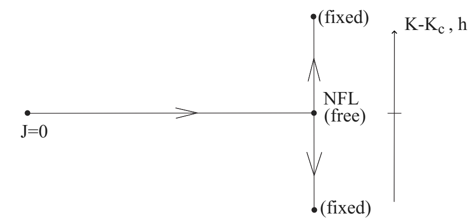

FIG. 4. Schematic flow diagram of the PH symmetric two-impurity Kondo model.

The conductance vanishes in both FL phases. At $K>K_c$ the system remains in its weak-coupling limit, corresponding to weakly transmitting tunnel junctions. At $K<K_c$ a Kondo-screened phase is developed and the two channels ψ_e and ψ_o participate in the screening of the combined spin-1 impurity. In the effective FL description both the even and odd channels acquire a phase shift of $\delta_e=-\delta_o=\pi/2$. The conductance vanishes as a result of *destructive interference* between even and odd channels: an incoming electron from the left lead $\psi_L^{\text{in}}=\frac{\psi_e^{\text{in}}+\psi_o^{\text{in}}}{\sqrt{2}}$ scatters into the outgoing state $\frac{\psi_e^{\text{out}}e^{2i\delta_e}+\psi_o^{\text{out}}e^{2i\delta_o}}{\sqrt{2}}=-\psi_L^{\text{out}}$ in the left lead, corresponding to full reflection. (The situation is reversed if a $\pi/2$ phase shift occurs only in one channel). We point out that when we include PH symmetry breaking the conductance is finite in the FL phases.

In the Hamiltonian [Eq. (2.9)] the condition $J_m \neq 0$ is required to mix the impurity singlet and triplet subspaces of the Hilbert space. At $J_m=0$ the transition at $K=K_c$ corresponds to a level crossing between those subspaces. We point out that this special situation occurs in our system for the symmetric point $t_1=t_2$ ($\theta=\pi/4$) and at zero flux. In this case, when $K<K_c$ the conductance is $G=\frac{2e^2}{h}$ since the odd channel is decoupled, $\delta_o=0$, and as a result of Kondo effect in the even channel $\delta_e=\pi/2$. This decoupling took place in several theoretical studies of parallel QDs with no magnetic flux; see, e.g., Ref. 19.

IV. UNIVERSAL CROSSOVER AS A FUNCTION OF INTERIMPURITY INTERACTION K

In Sec. III A we calculated the conductance at the critical value of the interimpurity exchange interaction $K=K_c$ and assuming PH symmetry. In this situation the system flows from weak coupling ($J=0$) to a NFL fixed point, corresponding to free BC in the Ising sector. At finite $|K-K_c|$ the system flows to another fixed points as illustrated in Fig. 4. Depending on the sign of $K-K_c$, those two states correspond to fixing the boundary spin in a semi-infinite Ising chain to point up or down. Note that whereas both the attractive and $J=0$ fixed points in Fig. 4 correspond to fixed-boundary condition in the Ising sector, they differ by the impurity spin states. The latter are decoupled and contribute to the ground state degeneracy only at the repulsive $J=0$ fixed point.

The crossover along the horizontal line in Fig. 4 is governed by the Kondo energy scale T_K. The universality of this crossover between fixed- and free-boundary conditions in the Ising sector, however, is spoiled by irrelevant operators emerging from other sectors of the theory.

The crossover along the vertical line in Fig. 4 is governed by an energy scale $T^* = c_1 \frac{(K-K_c)^2}{T_K}$, where c_1 is a numerical factor of O(1). It was argued in Ref. 10 that this crossover is in the universality class of the crossover from free- to fixed-boundary conditions in the Ising model, driven by a local magnetic field h at the boundary of a quantum Ising chain. The magnetic field h is linearly related to $K-K_c$,

$$T^* = c_1 \frac{(K-K_c)^2}{T_K} = h^2. \tag{4.1}$$

When $T^* \ll T_K$ we may safely ignore sectors other than the Ising sector in the low energy crossover. This mapping of the 2IKM to the boundary Ising model opens the possibility to calculate the full crossover formula for the conductance as a function of $h \sim \frac{K-K_c}{\sqrt{T_K}}$ at finite temperature due to exact solvability of the boundary Ising model.

A. Boundary Ising model

It is well known that the scaling limit of the two-dimensional classical Ising model at its bulk critical point is described by a free massless Majorana field theory. Here we consider the two-dimensional model with a boundary, which is equivalent to the quantum semi-infinite chain. After unfolding the model in the standard fashion[23] we obtain a left moving Majorana fermion on the infinite line,

$$H_{\text{Ising}} = \frac{1}{2}\int_{-\infty}^{\infty} dx \chi(x) i\partial_x \chi(x) + H_B, \quad H_B = h\sigma_B. \tag{4.2}$$

At $h=0$ this model corresponds to free BC, expressed by the continuity of the chiral Majorana fermion field $\chi(x)$ at $x=0$. h is an external magnetic field acting on the boundary spin σ_B only. Clearly $h=\pm\infty$ implies fixed BC. The boundary spin can be written as[27]

$$\sigma_B = i\chi(x=0)a. \tag{4.3}$$

Here a is an additional Majorana fermionic boundary degree of freedom which anticommutes with χ and satisfies $a^2 = 1/2$.

The bulk energy operator of the Ising model corresponds to a mass term $m\chi\bar{\chi}$, which is a product of a left- and a right-moving Majorana fields. Therefore the left moving factor of the energy operator, which is the field we refer to as the energy operator, is just the free Majorana fermion $\epsilon(x) \sim \chi(x)$ with dimension $\Delta=1/2$. Note that χ was introduced most naturally within free BC, while ϵ was introduced to represent free fermions at the fixed BC fixed point. Indeed, for free BC of the Ising model $\chi(x)$ is continuous and $\epsilon(x)$ undergoes a $\pi/2$ phase shift at $x=0$ [see Eq. (3.6)]. Hence,

$$\epsilon(x) = \text{sgn}(x)\chi(x). \tag{4.4}$$

B. Energy correlator at finite boundary field

In a bulk CFT a typical local operator is a product of left- and right-moving factors $\phi(x) = \phi_L(x)\phi_R(x)$ where we suppress the time variable. The Ising model has three primary bulk operators denoted $\mathcal{O}_\Delta$, $(\Delta = 1/2, 1/16, 0)$. In the presence of a boundary at $x=0$ one can formulate the theory in terms of left moving fields only, $\phi(x) = \phi_L(x)\phi_L(-x)$, $x>0$. For example, $\mathcal{O}_{1/2}(x) = \epsilon(x)\epsilon(-x)$. In particular, at $\tau=0$, $y=-x$ the correlator of the left-moving Ising fields ϵ at any h is related to the *one-point function* of the bulk energy operator of the boundary Ising model, $\langle \epsilon(x,0)\epsilon(-x)\rangle_h = \langle \mathcal{O}_{1/2}(x)\rangle_h$. The one-point function of the bulk energy operator was calculated using the integrability of the boundary Ising model with the result[27–29]

$$\langle \mathcal{O}_{1/2}(x)\rangle_h = \int_{-\infty}^{\infty} \frac{du}{2\pi} \frac{e^{2iux}}{1+e^{\beta u}} \frac{ih^2/2 - u}{ih^2/2 + u}. \tag{4.5}$$

Here $\beta = T^{-1}$ is the inverse temperature. More generally consider the correlation function $\mathcal{C}_h(x,y,\tau) = \langle \epsilon(x,\tau)\epsilon(y,0)\rangle_h$. Consider a perturbative calculation of $\mathcal{C}_h(x,y,\tau)$ in H_B. It can be shown[26] that (i) the correction vanishes for $xy>0$, (ii) for $xy<0$ the correction is a function of $z = \tau + i(x-y)$. This implies that we can *analytically continue* the one-point function to find $\mathcal{C}_h(x,y,\tau)$,

$$\mathcal{C}_h(x,y,\tau) = \langle \mathcal{O}_{1/2}(x)\rangle_h|_{x\to -iz}, \quad x>0, y<0. \tag{4.6}$$

For $x<0$, $y>0$ one can use $\mathcal{C}_h(x,y,\tau) = -\mathcal{C}_h(y,x,-\tau)$, where the $-$ sign arises from the fermionic nature of ϵ.

C. Direct calculation of the energy correlator

For the present problem the desired correlator can be computed directly as will be done in this section. We turn to a calculation of the Majorana Green's function (GF) $\mathcal{G}(\tau,x,y) = -\langle \chi(x,\tau)\chi(y,0)\rangle$ at finite h and temperature $T = \beta^{-1}$. From Eq. (4.4), the energy correlator is

$$\langle \epsilon(x,\tau)\epsilon(y,0)\rangle_h = -\mathcal{G}(\tau,x,y)\text{sgn}(xy). \tag{4.7}$$

For $h=0$, $\mathcal{G}(\tau,x,y)$ is a free fermion GF,

$$\mathcal{G}^{(0)}(\tau,x,y) = \frac{1}{2\pi} \frac{-\pi/\beta}{\sin\left\{\frac{\pi}{\beta}[\tau + i(x-y)]\right\}} = \frac{1}{\beta}\sum_n e^{-i\omega_n \tau} \times \mathcal{G}^{(0)}(i\omega_n,x,y) = \frac{i}{\beta}\sum_n e^{-i\omega_n[\tau + i(x-y)]} \times [\theta(-\omega_n)\theta(x-y) - \theta(\omega_n)\theta(y-x)],$$

where $\omega_n = \frac{\pi}{\beta}(1+2n)$. Since the interaction in Eq. (4.3) is quadratic in fermion fields, we may sum up the perturbation series in the boundary magnetic field exactly, giving

$$\mathcal{G}(i\omega_n,x,y) = \mathcal{G}^{(0)}(i\omega_n,x,y) + h^2 \mathcal{G}^{(0)}(i\omega_n,x,0)\mathcal{G}_a(i\omega_n) \times \mathcal{G}^{(0)}(i\omega_n,0,y). \tag{4.8}$$

Here $\mathcal{G}_a(i\omega_n) = -\int_0^\beta d\tau e^{i\omega_n \tau}\langle a(\tau)a\rangle$ is the a propagator. When a is decoupled, its propagator is given by $\mathcal{G}_a^{(0)}(i\omega_n) = (i\omega_n)^{-1}$. Equation (4.8) becomes exact when $\mathcal{G}_a(i\omega_n)$ is calculated to infinite order in h. This is accomplished by the self-energy $\Sigma_a(i\omega_n)=h^2\mathcal{G}^{(0)}(i\omega_n,0,0)=-ih^2\,\mathrm{sgn}(\omega_n)/2$. Thus

$$\mathcal{G}_a(i\omega_n)=[i\omega_n+ih^2\,\mathrm{sgn}(\omega_n)/2]^{-1}. \tag{4.9}$$

Plugging this result in Eq. (4.8) yields the result

$$\mathcal{G}(i\omega_n,x,y)=\mathcal{G}^{(0)}(i\omega_n,x,y)+ie^{\omega_n(x-y)}\sum_{s=\pm1}\theta(s\omega_n)\times\theta(sy)\theta(-sx)\frac{h^2}{\omega_n+h^2\,\mathrm{sgn}(\omega_n)/2}. \tag{4.10}$$

When $xy>0$ there is no dependence on h. To compare Eq. (4.10) in the nontrivial region $xy<0$ with the result obtained by analytic continuation of the one-point function of the energy operator [Eq. (4.5)], we write the Fourier transform of Eq. (4.10) into

$$\mathcal{G}(\tau,x,y)=\int_{-\infty}^{\infty}\frac{du}{2\pi}\frac{e^{u[\tau+i(x-y)]}}{1+e^{\beta u}}\frac{ih^2\,\mathrm{sgn}(x-y)/2-u}{ih^2\,\mathrm{sgn}(x-y)/2+u},$$

valid for $x\cdot y<0$. One arrives at the same result using Eqs. (4.5)–(4.7). In this notation the integration variable u is related to the momentum of the particles used in the form factor method.

D. Finite temperature conductance

At finite temperature the conductance is obtained by analytic continuation,

$$G=\lim_{L\to\infty}\lim_{\omega\to0}\frac{ie^2}{\hbar\omega(2L)^2}\int_{-L}^{L}dx\int_{-L}^{L}dy\,\mathrm{sgn}(xy)\times\int_{-\beta/2}^{\beta/2}d\tau e^{-i\nu_n\tau}\langle j_o(x,\tau)j_o(y,0)\rangle|_{i\nu_n\to\omega+i0^+}, \tag{4.11}$$

where $\nu_n=\frac{2\pi n}{\beta}$. For $h=0$ the finite temperature odd current correlator, $\langle j_o(x,\tau)j_o(y,0)\rangle_{\text{free}}$, is given by Eq. (3.7) where $\langle j_o(x,\tau)j_o(y,0)\rangle_{J=0}=-\frac{1}{\beta^2\sin^2\{\frac{1}{\beta}[\tau+i(x-y)]\}}$. At finite h we use Eq. (3.3) and $\langle(j^f)^z(x,\tau)(j^f)^y(y)\rangle=0$, leading to

$$\langle j_o(x,\tau)j_o(y,0)\rangle_h=4\cos^2\phi_m\langle(j^f)^z(x,\tau)(j^f)^z(y)\rangle+4\sin^2\phi_m\langle(j^f)^y(x,\tau)(j^f)^y(y)\rangle.$$

Using the Bose-Ising representation of the flavor currents, given in Table I, and Eq. (4.7), we obtain

$$\langle j_o(x,\tau)j_o(y,0)\rangle_h=-4\cos^2\phi_m[\mathcal{G}^{(0)}(\tau,x,y)]^2-4\sin^2\phi_m\,\mathrm{sgn}(xy)\mathcal{G}^{(0)}(\tau,x,y)\mathcal{G}(\tau,x,y).$$

Compared to free BC, the odd current correlator obtains an additional term,

$$\langle j_o(x,\tau)j_o(y,0)\rangle_h=\langle j_o(x,\tau)j_o(y,0)\rangle_{\text{free}}-2\sin^2\phi_m\frac{\mathcal{G}(\tau,x,y)-\mathcal{G}^{(0)}(\tau,x,y)}{\beta\sin\left\{\frac{\pi}{\beta}[\tau+i(x-y)]\right\}}. \tag{4.12}$$

The first term contributes $\frac{2e^2}{h}\sin^2\phi_m$ to the conductance. The second term is nonvanishing only for $xy<0$, as can be seen from Eq. (4.10). Note that $\mathcal{G}(\tau,x,y)-\mathcal{G}^{(0)}(\tau,x,y)\to_{h\to\infty}-2\mathcal{G}^{(0)}(\tau,x,y)\theta(-xy)$, hence

$$\langle j_o(x,\tau)j_o(y,0)\rangle_{h\to\infty}=\langle j_o(x,\tau)j_o(y,0)\rangle_{\text{fixed}}$$

as expected. At finite T and h the contribution of the second term to the conductance is given by $2\,\mathrm{Re}\,G_1$, where

$$G_1=\lim_{L\to\infty}\lim_{\omega\to0}\frac{-ie^2}{\hbar\omega(2L)^2}\int_{-L}^{0}dx\int_0^L dy\langle j_o(x)j_o(y)\rangle^{(1)}_{i\nu_n\to\omega+i0^+},$$

where $\langle j_o(x)j_o(y)\rangle^{(1)}=\langle j_o(x)j_o(y)\rangle_h-\langle j_o(x)j_o(y)\rangle_{\text{free}}$. This correlator can be expressed as a Matsubara sum,

$$\langle j_o(x<0)j_o(y>0)\rangle^{(1)}_{i\nu_n}=\frac{4\sin^2\phi_m}{\beta^2}\int_{-\beta/2}^{\beta/2}d\tau e^{i\nu_n\tau}\times\sum_{m,l}\theta(\omega_m)\theta(\omega_l)e^{-i(\omega_m+\omega_l)(\tau+i(x-y))}\frac{ih^2}{i\omega_l+ih^2/2}=\frac{\sin^2\phi_m}{\pi^2}\frac{(2\pi)^2}{\beta}e^{\nu_n(x-y)}2\sum_{l=0}^{n-1}\frac{\beta h^2}{2\pi(1+2l)+\beta h^2}.$$

The sum is evaluated as an analytic function of $\nu_n=\frac{2\pi n}{\beta}$ in terms of the digamma function (the logarithmic derivative of the gamma function) $\psi(z)=d\log\Gamma(z)/dz$,

$$\sum_{l=0}^{n-1}\frac{4\pi}{2\pi(1+2l)+\beta h^2}=\psi\left(\frac{1}{2}+\frac{\beta h^2}{4\pi}+\frac{\beta\nu_n}{2\pi}\right)-\psi\left(\frac{1}{2}+\frac{\beta h^2}{4\pi}\right).$$

Performing the analytic continuation $i\nu_n\to\omega+i0^+$, sending $\omega\to0$, and performing the spatial integrations we obtain

$$G/G_0=1-F[T/T^*],\qquad F[t]=\frac{1}{4\pi t}\mathrm{Re}\,\psi_1\left(\frac{1}{2}+\frac{1}{4\pi t}\right), \tag{4.13}$$

where $\psi_1(z)=d^2\log\Gamma(z)/dz^2$ is the trigamma function and G_0 and T^* are given in Eqs. (3.9) and (4.1). The scaling function $F[x]$ has the properties $F[0]=1$ and $F[\infty]=0$. A signature of a NFL is the existence of relevant operators in the Hamiltonian with scaling dimension $\Delta<1$. The QD setup discussed here allows us to observe that the interimpurity interaction is such a relevant perturbation with $\Delta=1/2$. According to Eq. (4.13) the crossover from $G\sim G_0$ to insulating FL state as a function of $K-K_c$ occurs at a value of $|K-K_c|$ which scales with temperature as $T^{1/2}$.

E. Conductance in the model of Zaránd *et al.* (Ref. 17)

We pause here to comment on an application of the Ising model with boundary magnetic field for a different double QD model proposed by Zaránd *et al.*[17] as a realization of the 2IKM. We will show that in this system the full crossover of the conductance as a function of K in the PH symmetric point can be expressed in terms of the one-point function of the spin operator of the boundary Ising model.

Consider a modified QD system with an additional lead B coupled only to S_2 as in Fig. 5. Transport takes place between the left and right leads, where lead B acts as a screen-

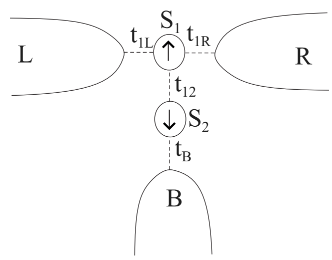

FIG. 5. Schematic description of the double QD proposed by Zaránd *et al.* (Ref. 17) as a realization of the 2IKM.

ing channel for spin S_2. The analysis of Sec. III goes through, and the conductance is given by Eq. (3.1), where the odd current is still written as $j_o(x)=\psi^{\dagger j\alpha}(\tau^z)^j_j\psi_{j\alpha}$, $j=1,2=L,R$. The next step in Sec. III was to rewrite $j_o(x)$ in the basis ψ' which is natural in the representation of the 2IKM Hamiltonian. For the present system this basis is

$$\psi_1' = \frac{t_{1L}\psi_L + t_{1R}\psi_R}{\sqrt{t_{1L}^2 + t_{1R}^2}}, \quad \psi_2' = \psi_B.$$

A third fermion $\psi_3'=\frac{-t_{1R}\psi_L+t_{1L}\psi_R}{\sqrt{t_{1L}^2+t_{1R}^2}}$ is decoupled from the impurities. We specialize to $t_{1L}=t_{1R}$. In this basis the correlator $\langle j_o(x,\tau)j_o(y,0)\rangle$ occurring in the Kubo formula factorizes into the product of GFs for ψ_1' and ψ_3', where the latter is a free fermion GF. For the GF of ψ_1' we use the bosonization formula [Eq. (3.4)], where the only factor which is sensitive to the critical point is the Ising spin operator, leading to

$$\langle j_o(x,\tau)j_o(y,0)\rangle = -\frac{1}{\pi^2}\frac{\langle\sigma(x,\tau)\sigma(y,0)\rangle}{[\tau+i(x-y)]^{2-1/8}}. \quad (4.14)$$

For finite h and T Eqs. (4.11) and (4.14) express the conductance in terms of the two-point function for the chiral spin operator at finite magnetic field h. Following the analysis leading to Eq. (4.6), the two-point function for the chiral spin operator is related to the one-point function of the bulk spin operator by analytic continuation,

$$\langle\sigma(x,\tau)\sigma(y,0)\rangle_h = \langle\mathcal{O}_{1/16}(x)\rangle_h|_{x\to -iz}, \quad x>0, y<0.$$

The calculation of the one-point function of the Ising spin at finite magnetic field h was addressed using integrability and the form factor method.[27–29] Different than the case of the energy operator, a closed expression for $\langle\mathcal{O}_{1/16}(x)\rangle_h$ is not available. In the limiting cases $h=0$ and $h=\pm\infty$ CFT methods can be used.[17] For BC obtained by fusion with operator a, Eq. (3.5) gives (with $\Delta=1/16$)

$$\langle\sigma(x,\tau)\sigma(y,0)\rangle = \begin{cases} \dfrac{1}{(\tau+i(x-y))^{1/8}}, & xy>0 \\ \dfrac{S_{(1)}}{(\tau+i(x-y))^{1/8}}, & xy<0, \end{cases} \quad (4.15)$$

where $S_{(1)}=\frac{S_a^{1/16}/S_0^{1/16}}{S_a^0/S_0^0}$. It is easy to calculate the conductance with Eq. (4.15), with the result $G=\frac{e^2}{h}(1-S_{(1)})$. At weak coupling $J=0$ $(a=0)$ we have $S_{(1)}=1$, $G=0$. This is also the result for the BC obtained by starting at the QCP and setting $K>K_c$ (local singlet phase). At the QCP $(a=1/16)$ we have $S_{(1)}=0$, $G=e^2/h$. In the Kondo-screened phase $(a=1/2)$ we have $S_{(1)}=-1$, $G=2e^2/h$. We leave for a future work to apply Eq. (4.14) in order to interpolate between those values of G at finite temperature and $h\propto(K-K_c)$. The additional difficulty for this system arises due to the presence of the σ GF rather than the ϵ GF.

V. UNIVERSAL CROSSOVER AT FINITE POTENTIAL SCATTERING

Until here we assumed PH symmetry and emphasized that the crossover is in the universality class of the boundary Ising model. Now we shall consider the more general situation with potential scattering (PS). We will see that the Ising and charge $SU(2)_1$ sectors of the theory are coupled. However this coupling can be written in a simple quadratic form in the Majorana SO(8) representation that will be introduced below.

It is convenient to write H_{PS}, defined in Eq. (2.2), in the ψ' basis [defined in Eq. (2.12)],

$$H_{PS} = \frac{V_L^L+V_R^R}{2}(\psi_1'^\dagger\psi_1'+\psi_2'^\dagger\psi_2') + \text{Re}\, V_L^R(\psi_1'^\dagger\psi_2'+\psi_2'^\dagger\psi_1') + V_A(\psi_1'^\dagger\psi_1'-\psi_2'^\dagger\psi_2') + V_B i(\psi_1'^\dagger\psi_2'-\psi_2'^\dagger\psi_1'), \quad (5.1)$$

where

$$V_A = \frac{V_L^L-V_R^R}{2}\cos\phi_m - \text{Im}\, V_L^R\sin\phi_m,$$

$$V_B = \frac{V_L^L-V_R^R}{2}\sin\phi_m + \text{Im}\, V_L^R\cos\phi_m. \quad (5.2)$$

In the parity-symmetric case $V_A=V_B=0$.

At the QCP the PS terms describing charge transfer between channels ψ_1' and ψ_2' generate relevant perturbations.[10] To see this consider their Bose-Ising representation (using Table I),

$$\psi_1'^\dagger\psi_2' + \text{H.c.} \sim (h_1)^\dagger\tau^z(h_2)\epsilon,$$

$$i\psi_1'^\dagger\psi_2' + \text{H.c.} \sim (h_1)^\dagger(h_2)\epsilon. \quad (5.3)$$

At the nontrivial fixed point the energy operator ϵ "disappears" by double fusion; hence one obtains two relevant boundary operators $(h_1)^\dagger\tau^z(h_2)$ and $(h_1)^\dagger(h_2)$, with dimension $\Delta=1/2$. In the parity-symmetric case only the first operator is allowed. These relevant operators have the dimension of a free fermion. Following Gan[11] a fermion representation emerges naturally in the SO(8) representation that we shall introduce in Sec. V A. In order for these relevant operators to have bosonic statistics, in the SO(8) representation indeed they are written as a product of a bulk fermion with a local fermion with dimension $\Delta=0$, which can be associated with a leftover impurity degree of freedom. On the other hand the intrachannel PS terms lead to marginal operators at the QCP,

$$\psi_1'^\dagger\psi_1' \pm \psi_2'^\dagger\psi_2' \sim I_1^z \pm I_2^z. \quad (5.4)$$

A. Fixed point Hamiltonian in SO(8) representation

Following Ref. 30 we bosonize the original theory and introduce four left moving bosonic fields: $\psi'^{\dagger j\alpha}\psi'_{j\alpha} := \frac{1}{2\pi}\partial_x\phi_{j\alpha}$. In terms of the bosons we can write the fermions as $\psi'_{j\alpha}\sim F_{j\alpha}e^{-i\phi_{j\alpha}}$. The Klein factors $F_{i\alpha}$ take care of our sign convention required for products of exponentials of bosonic fields. They satisfy[31]

$$[F_\mu,N_\nu]=\delta_{\mu\nu}F_\mu, \quad \{F_\mu,F_\nu^\dagger\}=2\delta_{\mu\nu},$$

$$(F_\mu F_\mu^\dagger=F_\mu^\dagger F_\mu=1), \quad \{F_\mu,F_\nu\}=0, \tag{5.5}$$

and $[F_\mu,\phi_\nu]=0$, where $\mu,\nu=\{i,\alpha\}$ and N_μ is the fermion number of species μ.

Subsequently four linear bosonic combinations are defined, corresponding to charge, spin, flavor, and difference of spin between the flavors,

$$\phi_c=\frac{1}{2}\sum_{j\alpha}\phi_{j\alpha}, \quad \phi_s=\frac{1}{2}\sum_{j\alpha}(\sigma^z)^\alpha_\alpha\phi_{j\alpha},$$

$$\phi_f=\frac{1}{2}\sum_{j\alpha}(\tau^z)^j_j\phi_{j\alpha}, \quad \phi_X=\frac{1}{2}\sum_{j\alpha}(\tau^z)^j_j(\sigma^z)^\alpha_\alpha\phi_{j\alpha}.$$

Since the exponents of these new bosons have dimension 1/2, we define new fermions $\psi_A\sim F_A e^{-i\phi^A}$, $A=c,s,f,X$. The new Klein factors satisfy Eq. (5.5) with $\mu,\nu=c,s,f,X$. To fix a convention we define[31]

$$F_X^\dagger F_s^\dagger=F_{1\uparrow}^\dagger F_{1\downarrow}, \quad F_X F_s^\dagger=F_{2\uparrow}^\dagger F_{2\downarrow}, \quad F_X^\dagger F_f^\dagger=F_{1\uparrow}^\dagger F_{2\uparrow}.$$

The free part of the Hamiltonian can be written equivalently in bosonic or fermionic form,

$$H_0=\sum_A\int\frac{dx}{2\pi}(\partial_x\phi_A)^2=\sum_A\int dx\,\psi_A^\dagger(i\partial_x)\psi_A.$$

Taking the real and imaginary parts of those fermions we obtain eight Majorana fermions,

$$\chi_1^A=\frac{\psi_A^\dagger+\psi_A}{\sqrt{2}}, \quad \chi_2^A=\frac{\psi_A^\dagger-\psi_A}{\sqrt{2}i}.$$

One can establish a connection between the descriptions of the 2IKM in terms of $\mathrm{SU}(2)_1^{\mathrm{charge1}}\times\mathrm{SU}(2)_1^{\mathrm{charge2}}\times\mathrm{SU}(2)_2^{\mathrm{spin}}\times\mathcal{Z}_2$ with eight Majorana fermions. The two $\mathrm{SU}(2)_1$ groups can be represented in terms of two bosons $\frac{\phi_c\pm\phi_f}{\sqrt{2}}$. The $\mathrm{SU}(2)_2^{\mathrm{spin}}$ current $\vec{j}^s=\frac{1}{2}\psi'^{\dagger i\alpha}\vec{\tau}_\alpha^\beta\psi'_{i\beta}$ has the representation $(j^s)^z=\psi_s^\dagger\psi_s$, $(j^s)^+=\sqrt{2}\chi_1^X\psi_s^\dagger$. Of particular interest for the present work, the flavor current [Eq. (3.2)] has the representation (see Table I)

$$(j^f)^z=\psi_f^\dagger\psi_f, \quad (j^f)^+=-\sqrt{2}i\psi_f^\dagger\chi_2^X. \tag{5.6}$$

The Ising fermion χ can be identified with χ_2^X. In fact, the nontrivial BC involves only one out of the eight Majorana fermions, reading $\chi_2^X(0^-)=-\chi_2^X(0^+)$. For a description of the physics relative to the nontrivial fixed point it is convenient to work with the continuous Ising fermion field,

$$\chi(x)=\mathrm{sgn}(x)\chi_2^X(x)=\epsilon(x)\mathrm{sgn}(x). \tag{5.7}$$

Using Eq. (4.3), in the PH symmetric case the relevant operator can be written as

$$\sigma_B=i\chi(x=0)a=i[\mathrm{sgn}(x)\chi_2^X(x)]_{x=0}\cdot a. \tag{5.8}$$

Now consider the non-PH symmetric case. From the SO(8) representation of the flavor current [Eq. (5.6)], the two PS terms in Eq. (5.3), $(j^f)^{x,y}$, are written in the trivial fixed point as $i\chi_2^X\chi_{1,2}^f$. CFT methods tell us that the operators at the QCP are obtained from the operators at the trivial fixed point by double fusion with the spin operator of the Ising model. Having identified χ_2^X with the Ising fermion, double fusion gives $\chi_2^X\to 1+\chi_2^X$. To obtain the correct bosonic statistics we argue that this fusion rule should be modified to

$$\chi_2^X\to a+\chi_2^X,$$

where a is the local fermion appearing in Eq. (5.8). Hence the relevant PS operators at the QCP are

$$(h_1)^\dagger\tau^z(h_2)\sim i\chi_1^f a,$$

$$(h_1)^\dagger(h_2)\sim i\chi_2^f a. \tag{5.9}$$

Thus, a couples the Ising sector with the charge sectors. The main argument in favor of this form is obtained by considering the self-correlation function of the relevant operators, e.g.,

$$\langle(h_1^\dagger h_2)(\tau)(h_1^\dagger h_2)\rangle\sim\mathcal{G}^{(0)}(\tau,0,0)\mathcal{G}_a(\tau),$$

at the PH symmetric point. Fourier transforming Eq. (4.9) for $\mathcal{G}_a(i\omega_n)$ we can deduce the behavior of $\mathcal{G}_a(\tau)$: in the limit $\tau\ll h^2(\tau\gg h^2)$, the correlator $\mathcal{G}_a(\tau)$ goes like $\tau^0(\tau^{-1})$. This implies that in these two limits the correlator $\langle(h_1^\dagger h_2)(\tau)(h_1^\dagger h_2)\rangle$ goes like $\tau^{-1}(\tau^{-2})$, respectively, as expected from an operator with scaling dimension $\Delta=\frac{1}{2}(1)$. This scaling behavior is obtained relying on the fact that a contains the information about the crossover. It explains why a, and not some other decoupled local operator, should be coupled to χ_1^f and χ_2^f in Eq. (5.9). On the contrary, the presence of an additional decoupled local operator at the QCP is ruled out as inconsistent with the ground state degeneracy. Away from the PH symmetric point, the local operator a becomes also sensitive to the deviation from the QCP due to potential scattering and $\mathcal{G}_a$ is modified relative to Eq. (4.9).

Setting together Eqs. (5.8) and (5.9), the correction to the fixed point Hamiltonian in SO(8) representation is

$$\delta H=i[\lambda_1\chi_2^X(x)\mathrm{sgn}(x)+\lambda_2\chi_1^f(x)+\lambda_3\chi_2^f(x)]a|_{x=0}, \tag{5.10}$$

with

$$\lambda_1=c_1\frac{K-K_c}{\sqrt{T_K}},$$

$$(\lambda_2,\lambda_3)=c_2\sqrt{T_K}\nu(\mathrm{Re}\,V_L^R,V_B), \tag{5.11}$$

where V_L^R and V_B are given in Eqs. (2.8) and (5.2) and c_1 and c_2 are constants of O(1). This estimate of (λ_2,λ_3) will be justified below; as we shall see, based on the dimension $\Delta=1/2$ of the three relevant operators in Eq. (5.10) we obtain the crossover energy scale,

$$T^*=\lambda_1^2+\lambda_2^2+\lambda_3^2\equiv\lambda^2. \tag{5.12}$$

To estimate λ_2 and λ_3 we consider the renormalization group flow of the interchannel potential scattering operators $(\psi'^{\dagger 1\alpha}\psi'_{2\alpha}\pm\text{H.c.})$. In the presence of those operators the flow to the QCP stops at energy scale T^*_{LR}. To estimate T^*_{LR} we consider the renormalization of these operators in the perturbative regime at energy scales $D\gg T_K$ and then in the nonperturbative regime at energy scales $D\ll T_K$, respectively. (A related calculation for the two channel Kondo model appears in Ref. 32.) We assume that $K=K_c$. At the initial scale $D_0\gg T_K$ the dimensionless bare values of these PS operators are $\Delta_0=\nu\,\text{Re}\,V_L^R$ and $\Delta_0'=\nu V_B$ [see Eq. (5.1)]. We assume $\Delta_0,\Delta_0'\ll 1$. Since in the weak-coupling regime potential scattering does not renormalize, we have

$$\Delta(T_K)\sim\Delta_0,\quad \Delta'(T_K)\sim\Delta_0'.$$

These can be viewed as the initial values of the coupling constants of the relevant perturbations $(h_1)^\dagger\tau^z(h_2)$ and $(h_1)^\dagger(h_2)$, respectively. Since these operators have dimension 1/2, the dependence of their coupling constants on $D\ll T_K$ is described by

$$\frac{\Delta(D)}{\Delta(T_K)}\sim\frac{\Delta'(D)}{\Delta'(T_K)}\sim\left(\frac{T_K}{D}\right)^{1/2}.$$

The condition $\max\{\Delta(T^*_{LR}),\Delta'(T_{\text{PS}})\}\sim 1$ gives the estimate

$$T^*_{LR}\sim\max\{T_K\Delta_0^2,T_K(\Delta_0')^2\}. \tag{5.13}$$

A more precise estimate would take into account higher order terms in the β function for Δ,Δ'. However, we expect that this would only change our estimate of T^*_{LR} by logarithmic factors.

Identifying T^*_{LR} with $\lambda_2^2+\lambda_3^2$ in Eq. (5.12) gives the estimate for λ_2 and λ_3 given in Eq. (5.11). Under the condition $\Delta_0,\Delta_0'\ll 1$ one has a wide energy range $T^*_{LR}\ll D\ll T_K$ for the observation of the QCP. This can occur in a certain parameters regime, as we discuss in Sec. VI A.

We point out that our estimate for the energy scale T^*_{LR} [Eq. (5.13)], which agrees with Ref. 17, is inconsistent with that of Sakai and Shimizu,[21] who studied the 2IKM with finite transfer matrix between the impurities using numerical renormalization group. This discrepancy requires further investigation.

B. Linear conductance with potential scattering

We generalize the linear conductance calculation of Sec. IV for finite potential scattering. Using Eqs. (3.3) and (5.6) the odd current operator is

$$j_o=2i\chi_2^f[\cos\phi_m\chi_1^f+\sin\phi_m\chi_2^X\,\text{sgn}(x)]. \tag{5.14}$$

The operator a is now coupled to three free Majorana fields, and its GF [Eq. (4.9)] generalizes to $\mathcal{G}_a(i\omega_n)=[i\omega_n+i\lambda^2\,\text{sgn}(\omega_n)/2]^{-1}$, where λ^2 is defined in Eq. (5.12). Similarly,

$$-\langle\chi_i(x)\chi_j(y)\rangle_{i\omega_n}=\mathcal{G}^{(0)}(i\omega_n,x,y)\delta_{ij}+h_ih_j\delta\mathcal{G}(i\omega_n,x,y),$$

where

$$\delta\mathcal{G}(i\omega_n,x,y)=\mathcal{G}^{(0)}(i\omega_n,x,0)\mathcal{G}_a(i\omega_n)\mathcal{G}^{(0)}(i\omega_n,0,y).$$

Generalizing Eq. (4.12) we obtain the odd current correlator,

$$\begin{aligned}\langle j_o(x,\tau)j_o(y,0)\rangle_{\lambda_1,\lambda_2,\lambda_3}&=\langle j_o(x,\tau)j_o(y,0)\rangle_{\text{free}}\\&\quad+[\sin^2\phi_m(\lambda_1^2+\lambda_3^2)\\&\quad-\cos^2\phi_m(\lambda_2^2+\lambda_3^2)]\\&\quad\times 4\mathcal{G}^{(0)}(i\omega_n,x,y)\delta\mathcal{G}(i\omega_n,x,y).\end{aligned}$$

As a result the conductance has the scaling form

$$G/G_0=1-F[T/T^*]\frac{\sin^2\phi_m(\lambda_1^2+\lambda_3^2)-\cos^2\phi_m(\lambda_2^2+\lambda_3^2)}{\lambda^2\sin^2\phi_m}. \tag{5.15}$$

We see that the conductance at the free fixed point ($\lambda=0$) is still given by $G_0=\frac{2e^2}{h}\sin^2\phi_m$. At $\lambda\to\infty$ the Fermi-liquid conductance is

$$G_{\text{FL}}=\frac{2e^2}{h}\frac{(\lambda_2)^2+\cos^2\phi_m(\lambda_3)^2}{\lambda^2}. \tag{5.16}$$

We may rewrite Eq. (5.15) as

$$\frac{G-G_{\text{FL}}}{G_0-G_{\text{FL}}}=1-F[T/T^*].$$

C. Gan's theory and its relation to boundary Ising model

Gan[11] presented a solution of the 2IKM, constructing an effective Hamiltonian for a finite region in the phase diagram around the critical point by controlled projection. The effective Hamiltonian is solved exactly not only at the critical point but also for the surrounding Fermi-liquid phase. Excellent agreement was found with numerical renormalization group and CFT, in spite of the fact that the theory of Gan is not spin-SU(2) invariant. We shall substantiate the relation of Gan's theory to the CFT by showing explicitly that the operators at the critical point have the same form for both theories. In Sec. VI we will use this approach to calculate the nonlinear conductance.

Gan's theory uses the SO(8) representation, and the two-impurity spins turn into a local fermion d, where $\{d,d^\dagger\}=1$. Defining two Majorana fermions $a=\frac{d-d^\dagger}{\sqrt{2}i}$ and $b=\frac{d+d^\dagger}{\sqrt{2}}$, Gan's Hamiltonian in the PH symmetric case involves only the spin-flavor (X) sector and can be written as $H_G=H_G^{(0)}+\delta H_G$, where

$$H_G^{(0)}=\frac{1}{2}\int dx\chi_2^X i\partial_x\chi_2^X,$$

$$\delta H_G=2i\sqrt{T_K}\chi_2^X(0)b-i(K-K_c)ab. \tag{5.17}$$

We shall show that for energy scales $\ll T_K$ this coincides with the Ising model [Eq. (4.2)]. To see this suppose $K=K_c$ and consider a mode expansion,

$$\chi_2^X(x) = \sum_k^{\Lambda} (\varphi_k(x)\psi_k + \text{H.c.}), \quad b = \sum_k^{\Lambda} (u_k\psi_k + \text{H.c.}),$$

where $\{\psi_k, \psi_{k'}^\dagger\} = \delta(k-k')$, $\{\psi_k, \psi_{k'}\} = 0$, and where initially we choose $\Lambda \gg T_K$ as an ultraviolet cutoff. In the basis of ψ_k Gan's Hamiltonian is equal to $H = \Sigma_k \epsilon_k \psi_k^\dagger \psi_k$. One can obtain a Schrödinger's equation for the wave functions $\varphi_k(x)$ and u_k by equating the expansions of $[H_G, \chi_2^X(x)] = [H, \chi_2^X(x)]$ and $[H_G, b] = [H, b]$. One obtains

$$2i\sqrt{T_K}\delta(x)u_k + i\partial_x\varphi_k(x) = \epsilon_k\varphi_k(x),$$

$$-2i\sqrt{T_K}\varphi_k(0) = \epsilon_k u_k.$$

The solutions are $\varphi_k(x) \propto e^{ikx}[\theta(x)\varphi_k^{(+)} + \theta(-x)\varphi_k^{(-)}]$, $\varphi_k(0) = \frac{1}{2}(\varphi_k^{(+)} + \varphi_k^{(-)})$, $u_k = \frac{2}{i\epsilon_k}\sqrt{T_K}\varphi_k(0)$, $\varphi_k^{(-)}/\varphi_k^{(+)} = e^{2i\delta}$, $\tan\delta = \frac{2T_K}{\epsilon_k}$, and $\epsilon_k = -k$ (note that we work with left movers). While at $T_K = 0$ we have the BC $\chi_2^X(0^+) = \chi_2^X(0^-)$, we see from the wave function that the effect of the first boundary term in H_G is to modify this BC to $\chi_2^X(0^+) = -\chi_2^X(0^-)$ for energies $\ll T_K$. The key observation is that the following operator identity holds if one restricts the mode expansion of its left-hand side (LHS) and right-hand side (RHS) to energies below a cutoff $\Lambda \ll T_K$,

$$b = \frac{1}{\sqrt{T_K}}\chi_1(0), \tag{5.18}$$

where $\chi_1(x) = \chi_2^X(x)\text{sgn}(x)$. Physically this means that at energy scales below T_K the local operator b is absorbed into the field χ_2^X and changes its BC. Using the operator identity [Eq. (5.18)], we see that the term $\propto K - K_c$ in δH_G is equivalent to the boundary operator in the Ising model [Eq. (5.8)]. This establishes the connection between Gan's theory and the boundary Ising model arising from the CFT solution, showing that Gan's anisotropic theory describes correctly also the vicinity of the isotropic fixed point.

VI. CROSSOVER AT FINITE BIAS

Gan's formulation of the QCP in the SO(8) Majorana representation provides a direct way to calculate the nonlinear conductance at finite source drain voltage along the crossover from the NFL fixed point to the surrounding FL fixed points, including the PH symmetry breaking. Relegating the details of the calculation based on the Keldysh technique to the Appendix, our result is

$$G = G_0 + G_S F\left[\frac{T}{T^*}, \frac{eV}{T^*}\right] + G_A F'\left[\frac{T}{T^*}, \frac{eV}{T^*}\right],$$

$$F[t,v] = \frac{1}{4\pi t}\text{Re}\ \psi_1\left(\frac{1}{2} + \frac{1}{4\pi t} + \frac{iv}{2\pi t}\right),$$

$$F'[t,v] = \frac{1}{4\pi t}\text{Im}\ \psi_1\left(\frac{1}{2} + \frac{1}{4\pi t} + \frac{iv}{2\pi t}\right),$$

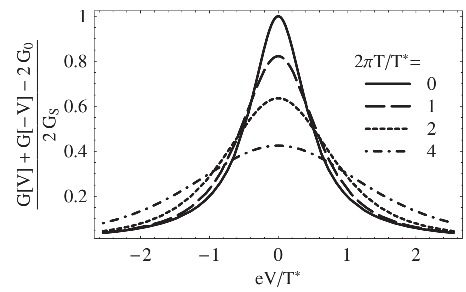

FIG. 6. Scaling function for the V-symmetric part of the nonlinear conductance.

$$\frac{G_S}{\frac{2e^2}{h}} = \frac{-\lambda_1^2\sin^2\phi_m + \lambda_2^2\cos^2\phi_m + \lambda_3^2(1-2\sin^2\phi_m)}{\lambda^2},$$

$$\frac{G_A}{\frac{2e^2}{h}} = \sin(2\phi_m)\lambda_3\frac{\lambda_2\sin\phi_m + \lambda_1\cos\phi_m}{\lambda^2}. \tag{6.1}$$

Here T^*, G_0, ϕ_m, and θ are given in Eqs. (5.12), (3.9), (2.11), and (2.7), respectively; $\psi_1(z)$ is defined below Eq. (4.13). This result is valid for $eV, T, T^* \ll T_K$. When $T^* \gg T, eV$ the system is in the FL state and the nonlinear conductance coincides with the linear conductance [Eq. (5.16)], $G_{\text{FL}} = G_0 + G_S$.

The scaling functions $F[t,v]$ and $F'[t,v]$ are symmetric and asymmetric in v, respectively (see Figs. 6 and 7). Having $G[V] \neq G[-V]$ is a signature of interactions since the Landauer noninteracting formula[22] leads to $G[V] = G[-V]$. This leads to a universal rectification effect. This rectification effect is odd under parity, $\lambda_3 \to -\lambda_3$. Note however that it does not have a well-defined transformation property with respect to $\Phi \to -\Phi$. To check the symmetry properties of our results we considered the two-impurity Anderson model for our model (Fig. 1) to first order in the (intradot) interaction U. While at $U=0$ we have $G[V] = G[-V]$, which follows from Landauer formula, to first order in U we get a finite $G[V] - G[-V]$. This asymmetric behavior of the conductance follows from an asymmetric dependence of the occupation of the dots on voltage. This simple limit gives the same symmetry properties of $G[V] - G[-V]$ compared to the QCP, namely, the rectification effect is odd under a parity, and does

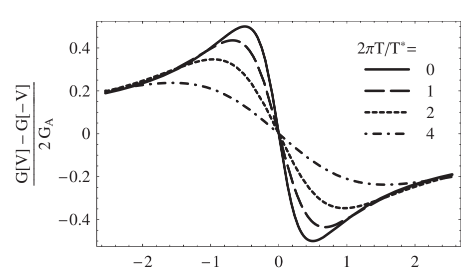

FIG. 7. Scaling function for the V-asymmetric part of the nonlinear conductance.

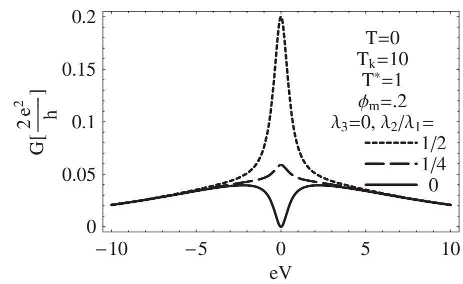

FIG. 8. Nonlinear conductance at $\lambda_3=0$ for $\lambda_2/\lambda_1=0,1/4,1/2$. The line shape consists of a narrow peak or dip structure of width T^*, superimposed on top of a wide peak of width T_K.

not have a well-defined symmetry property with respect to $\Phi \to -\Phi$.

At energy scales comparable to T_K the conductance has additional voltage and temperature dependence due to irrelevant operators at the QCP. The leading irrelevant operator is $H_{\mathrm{irr}}=T_K^{-1/2} i\partial_x \chi_1(x) a|_{x=0}$, with dimension $\Delta=3/2$.[9,10] In the proposed realization of the 2IKM of Zaránd *et al.*[17] it leads to the conductance correction $\delta G \propto \sqrt{\frac{T}{T_K}}$, characteristic of a NFL fixed point. However in the present system the irrelevant operator gives a nonzero correction only to fourth order, leading to $\delta G \propto (\frac{T}{T_K})^2$, as we outline below. The a-GF has an additional self-energy $\Sigma^R=-i\omega^2/T_K$,

$$G_a^R(\omega)=\frac{1}{\omega+iT^*/2} \to \frac{1}{\omega+i(T^*/2+\omega^2/T_K)}.$$

This has poles at $\omega=-iT_K\frac{1}{2}(1\pm\sqrt{1-\frac{2T^*}{T_K}})$. For $T^* \ll T_K$ we have

$$G_a^R(\omega) \cong \frac{1}{\omega+iT^*/2}-\frac{1}{\omega+iT_K}.$$

Qualitatively, the irrelevant correction at finite T_K has the same form of the fixed point conductance with $T^*/2 \to T_K$. Indeed at energy scales smaller than T^*, the latter has quadratic dependence on T/T^* and eV/T^*.[7] It should be pointed out that to fourth order in H_{irr} it is no longer consistent to disregard more irrelevant operators of dimension $\Delta=2$. However their inclusion leads only to the modification of the effective Kondo temperature in the corrections $(T/T_K)^2$ and $(eV/T_K)^2$.

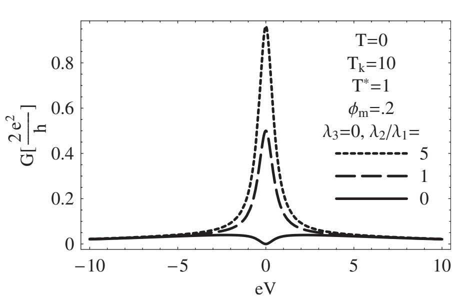

FIG. 9. Nonlinear conductance at $\lambda_3=0$ for $\lambda_2/\lambda_1=0,1,5$, reaching the unitary limit when the relevant perturbation is dominated by potential scattering, namely, $\lambda_2 \gg \lambda_1$.

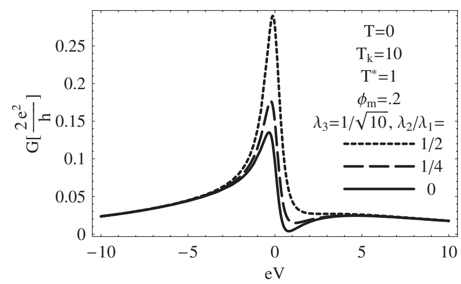

FIG. 10. Nonlinear conductance at finite $\lambda_3=1/\sqrt{10}$ for $\lambda_2/\lambda_1=0,1/4,1/2$, showing asymmetric features.

In Figs. 8 and 9 the conductance is plotted in the parity-symmetric case at $\lambda_3=0$ and zero temperature as a function of source drain voltage for different ratios λ_2/λ_1. The generic behavior of $G[V]$ consists of a wide peak of width T_K and height G_0, with a superimposed narrow structure (peak or dip) of width T^*, with height G_S (relative to the background G_0). Note that G_S is positive (negative) for $\lambda_1^2 \sin^2\phi_m < (>) \lambda_2^2 \cos^2\phi_m+\lambda_3^2(1-2\sin^2\phi_m)$, leading to a narrow peak (dip). When $\lambda_3=0$ and $\lambda_1 \tan\phi_m \ll \lambda_2$, Eq. (6.1) predicts a peak amplitude close to the unitary limit $2e^2/h$. For this case, we mention that when T_{LR}^* and $|K-K_c| \gtrsim T_K$, our results do not apply, and we expect a splitting of this peak as a function of V.[15,33] We can obtain this behavior on a qualitative level by going back to the high energy $E \sim T_K$ description with Eq. (5.17).

In Figs. 10 and 11 we plot the conductance under the same conditions except $\lambda_3=1/\sqrt{10}$ and $\lambda_3=1/\sqrt{2}$, respectively, showing asymmetric behavior. When $G_A>0$ ($G_A<0$) [defined in Eq. (6.1)], the slope of the conductance at $V=0$, $\frac{dG}{dV}|_{V=0}$, is negative (positive). The sign of G_A is changed under a parity transformation ($\lambda_3 \to -\lambda_3$), but it also depends on the sign of the combination $[\sin(2\phi_m)(\lambda_2 \sin\phi_m+\lambda_1\cos\phi_m)]$.

A. Observability

In this section we discuss the realizability of the critical point in real experiment. Dealing with a repulsive critical point, the first condition we are concerned with is the smallness of the relevant perturbations, $T^* \ll T_K$. Second, we shall list some marginal corrections.

In order to tune $K=K_c>0$, it is needed to reduce the ferromagnetic contribution K_{RKKY} [Eq. (2.3)] compared to

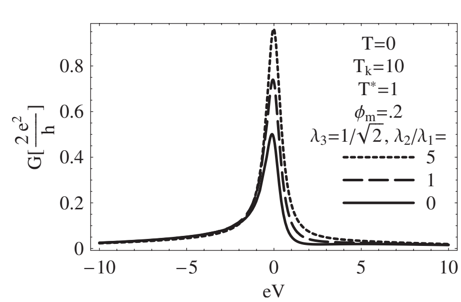

FIG. 11. Nonlinear conductance at finite $\lambda_3=1/\sqrt{2}$ for $\lambda_2/\lambda_1=0,1,5$.

K_{12} [Eq. (2.1)], either (i) by setting $\Phi=\pi$, which is sufficient in the ideal situation where the device is perfectly parity symmetric, or (ii) in the more generic and realistic case, where parity symmetry is only approximate, by creating *large asymmetry*,

$$t_2/t_1 \ll \frac{t_{12}}{U(\nu J)} \sim \sqrt{\frac{T_K}{U}}\frac{1}{\nu J}. \tag{6.2}$$

The limit $t_2=0$ corresponds to the series QD. In either case, using Eq. (2.1), the condition $K=K_c \sim T_K$ is achieved by tuning the interdot coupling $t_{12}\sim\sqrt{UT_K}$.

At $K=K_c$ ($\lambda_1=0$) Eqs. (5.11) and (5.12) give the crossover scale,

$$T^*_{LR} = T^*|_{K=K_c} = T_K[(\mathrm{Re}\ \nu V_L^R)^2 + (\nu V_B)^2],$$

where V_B and V_L^R are given in Eqs. (5.2) and (2.8), respectively. In the parity-symmetric case (i) we have $V_B=0$, and V_L^R is real and dominated by its second term in Eq. (2.8) because $\Phi=\pi$, leading to

$$T^*_{LR}/T_K \sim |\nu V_L^R|^2 \sim (\nu J)^2\frac{T_K}{U} \ll 1, \tag{6.3}$$

as required for the validity of the critical theory.

In the more realistic case (ii), on top of Eq. (6.2) we bound t_2/t_1 from below,

$$\sqrt{\frac{T_K}{U}} \sim t_{12}/U \lesssim t_2/t_1 \ll \sqrt{\frac{T_K}{U}},$$

such that V_L^R is dominated by the first term in Eq. (2.8). In addition we demand approximate parity symmetry,

$$\frac{|t_{1L}-t_{2L}|}{t_1} \ll \sqrt{T_K/U}(t_1/t_2), \quad \sin(\Phi_L-\Phi_R) \ll 1,$$

such that $|V_B| \ll \mathrm{Re}\ V_L^R$. Here $t_1=\frac{t_{1L}+t_{2L}}{2}$ and $t_2=\frac{t_{2L}+t_{1L}}{2}$. It leads to $T^*_{LR}/T_K \sim |\nu V_{LR}|^2 \sim (\nu J)^2(\frac{t_2}{t_1})^2 \ll 1$, as required for the validity of the critical theory.

Next we estimate the marginal corrections. Spin SU(2) symmetry is broken by the Zeeman energy $E_Z = g\mu_B B(S_1^z + S_2^z)$. This leads to a marginal operator[11] which reads in the Bose-Ising representation $\vec{\phi}\ \epsilon$. In GaAs QDs, the Zeeman energy is reduced due to a small g factor: for the experimental conditions in Ref. 1 T_K corresponds to a magnetic field of few tesla or equivalently to $\sim 10^3$ flux quanta in a area of $\mu\mathrm{m}^2$; for a magnetic field corresponding to $\Phi=\pi$ we have $\langle E_Z\rangle \sim 10^{-3}T_K$, leading to small marginal correction to the conductance.

Other marginal operators allowed at the QCP are the inter- and intrachannel PS [Eqs. (5.3) and (5.4)]. These operators have the SO(8) Majorana representations $\psi_1'^{\dagger}\psi_2' + \mathrm{H.c.} \sim i\chi_2^X\chi_1^f$, $i\psi_1'^{\dagger}\psi_2' + \mathrm{H.c.} \sim i\chi_2^X\chi_2^f$, $\psi_1'^{\dagger}\psi_1' \sim i\chi_1^c\chi_2^c + i\chi_1^f\chi_2^f$, and $\psi_2'^{\dagger}\psi_2' \sim i\chi_1^c\chi_2^c - i\chi_1^f\chi_2^f$, and can lead to corrections of $O(\nu^2J^2)$ to the conductance. However, the $T=V=0$ conduction peak at $K=K_c$ is expected to be unaffected, with $G=2e^2/h$. This follows because, except for $\psi_1'^{\dagger}\psi_1' + \psi_2'^{\dagger}\psi_2'$, which involves only the charge sector and does not affect transport, the other PS operators involve fields whose BC are strongly modified: due to Eq. (5.7), $\chi_2^X(0^+) = -\chi_2^X(0^-)$, and due to δH, Eq. (5.10), at energies $\ll T^*_{LR}$, we have $(\lambda_2\chi_1^f + \lambda_3\chi_2^f)(0^+) = -(\lambda_2\chi_1^f + \lambda_3\chi_2^f)(0^-)$. The evaluation of the original fields in terms of which the PS operators are written at the boundary gives zero, e.g., $\chi_2^X(0) = \frac{1}{2}[\chi_2^X(0^-) + \chi_2^X(0^+)] \to 0$.

Parity symmetry [Eq. (2.5)] of the Kondo interaction H_K is not required. This follows because in the PH symmetric case there are no relevant operators at the QCP that are odd under parity. Indeed the QCP has been observed numerically for a broken parity Kondo Hamiltonian.[17] With finite PS, we have considered explicitly the effect of the relevant parity-odd PS terms.

VII. CONCLUSIONS

We studied double quantum dots in the vicinity of the quantum critical point of the two-impurity Kondo model. In the PH symmetric model we used a mapping to the boundary Ising model with finite boundary magnetic field to calculate the finite temperature crossover of the conductance from the QCP to the stable fixed points. This method generalizes the CFT approach, which addresses only the vicinity of the fixed points. We used this method to relate the conductance of the proposed system of Zaránd *et al.*[17] to the one-point function of the magnetization operator in the boundary Ising model which can be calculated numerically.

Using the method developed by Gan, we solved the general and experimentally relevant case with potential scattering and found the nonlinear conductance at finite temperature along the multidimensional crossover from QCP to surrounding FL states. Compared to the series double QD, we found that in the general configuration the universal scaling function contains both symmetric and asymmetric terms in the source drain voltage, leading to a current rectification.

ACKNOWLEDGMENTS

We thank A. Aharony, D. Eigler, O. Entin-Wohlman, J. Folk, D. Goldhaber-Gordon, S. Loth, and J. Malecki for very helpful discussions. This work was supported by NSERC (E.S. and I.A.) and CIfAR (I.A.).

APPENDIX: CALCULATION OF THE NONLINEAR CONDUCTANCE USING KELDYSH GREEN'S-FUNCTION TECHNIQUE

We briefly recall basic concepts of the nonequilibrium formulation. Then the problem at hand will be addressed, and the calculation of the nonlinear conductance will be outlined.

One usually assumes that the system is in equilibrium at some initial time, taken here to be $t=-\infty$. A perturbation H_1 is turned on adiabatically in time, $H=H_0+e^{\eta t}H_1$ to drive the system out of equilibrium. The expectation value of an operator such as the current I is given by its trace in the Heisenberg picture at $t=0$ weighted by the initial distribution function,

$$\langle I\rangle = \mathrm{Tr}\{e^{-\beta H_0}u^\dagger(0,-\infty)\hat{I}u(0,-\infty)\},$$

where $u(t_0,t)=\mathcal{T}\exp(-i\int_{t_0}^{t}dt' H(t'))$ and $\mathcal{T}$ is the time ordering operator. In order to employ Wick's theorem, one transforms to the interaction picture, $\langle I\rangle=\mathrm{Tr}\{e^{-\beta H_0}u_I^\dagger(0,-\infty)\hat{I}_I u_I(0,-\infty)\}$, where $u_I(t_0,t)=\mathcal{T}\exp(-i\int_{t_0}^{t}dt'(H_1)_I(t'))$, and $\mathcal{O}_I(t)=e^{iH_0t}\mathcal{O}e^{-iH_0t}$. Following Keldysh, for a perturbative expansion of this quantity it is convenient to introduce four types of GFs,[34]

$$G^{11}(1,1') = -i\langle \mathcal{T}\chi(1)\chi(1')\rangle,$$

$$G^{12}(1,1') = G^{<}(1,1') = i\langle \chi(1')\chi(1)\rangle,$$

$$G^{21}(1,1') = G^{>}(1,1') = -i\langle \chi(1)\chi(1')\rangle,$$

$$G^{11}(1,1') = -i\langle \tilde{\mathcal{T}}\chi(1)\chi(1')\rangle.$$

Here $\tilde{\mathcal{T}}$ is the antitime ordering operator. It is convenient to consider an alternative set of GFs by defining the Keldysh GF matrix $\underline{G}=\begin{pmatrix} G^R & G^< \\ 0 & G^A\end{pmatrix}$, where

$$G^{R,A}(1,1') = \mp i\theta[\pm(t_1-t_1')]\langle\{\chi(1),\chi(1')\}_+\rangle.$$

Given a self-energy, $\underline{\Sigma}=\begin{pmatrix} \Sigma^R & \Sigma^< \\ 0 & \Sigma^A\end{pmatrix}$, the Keldysh GF matrix has the expansion

$$\underline{G}(\omega) = \underline{G}^{(0)}(\omega) + \underline{G}^{(0)}(\omega)\underline{\Sigma}(\omega)\underline{G}^{(0)}(\omega) + \cdots, \tag{A1}$$

where matrix multiplication in Keldysh space is understood and $A(\omega)=\int dt e^{i\omega t}A(t)$. This leads to the Dyson equation for the retarded/advanced components of $\underline{G}$,

$$G^R(\omega) = G^{R^{(0)}}(\omega) + G^{R^{(0)}}(\omega)\Sigma^R(\omega)G^R(\omega), \tag{A2}$$

and to the Keldysh equation,

$$G^< = G^R\Sigma^<G^A + (1+G^R\Sigma^R)G^{<(0)}(1+\Sigma^AG^A). \tag{A3}$$

We now apply this scheme to our problem with

$$H_0 = \sum_{j=1}^{2}\sum_{A=c,s,f,X}\frac{1}{2}\int_{-\infty}^{\infty}dx\chi_j^A i\partial_x\chi_j^A + \frac{eV}{2}\sum_\alpha (N_{L\alpha}-N_{R\alpha}),$$

$$H_1 = \delta H_G + i\lambda_2\chi_1^f a + i\lambda_3\chi_2^f a,$$

where $N_{i\alpha}=\int dx\psi^{\dagger i\alpha}\psi_{i\alpha}$ $(i=L,R)$. Here H_0 is the $J=0$ fixed point Hamiltonian, including the source drain voltage V, and δH_G is given in Eq. (5.17). It is more convenient to use δH_G, which includes the local b operator rather than the first term in δH, $i\lambda_1\chi_2^X(x)\mathrm{sgn}(x)a$. Both formulations should give the same result for energy scales $\ll T_K$, as we showed generally in Sec. V C. At $t=-\infty$ the system consists of two decoupled leads at equilibrium with different chemical potentials. It is convenient to make a change of basis, in which the operator $Y=\frac{1}{2}\sum_\alpha(N_{L\alpha}-N_{R\alpha})=\frac{1}{2}\int_{-\infty}^{\infty}dxj_o(x)$ is diagonal. Using Eq. (5.14) for j_o, we see that

$$Y = \int_{-\infty}^{\infty}dx\alpha^\dagger\alpha = i\int_{-\infty}^{\infty}dx\alpha_-\alpha_+,$$

where we defined new fermions α and β, $\alpha=\frac{\alpha_+-i\alpha_-}{\sqrt{2}}$, $\beta=\frac{\beta_+-i\beta_-}{\sqrt{2}}$, in terms of the four Majorana fermions $\alpha_\pm$ and $\beta_\pm$ given by

$$\alpha_+ = \chi_2^f, \quad \beta_+ = \chi_1^X,$$

$$\alpha_- = -(\cos\phi_m\chi_1^f + \sin\phi_m\chi_2^X),$$

$$\beta_- = (\sin\phi_m\chi_1^f - \cos\phi_m\chi_2^X).$$

We see that the voltage raises the chemical potential of the α fermions by eV, whereas the chemical potential for the β fermions remains zero. The system is at equilibrium at $t=-\infty$ since in this case the α and β fermion numbers are conserved. At $t>-\infty$, H_1 leads to the current operator $\hat{I}=i[Y,H_1]$ which drives the system out of equilibrium and is given by

$$\hat{I} = -2i\sqrt{T_K}\sin\phi_m\alpha_+(0)b - i\lambda_2\cos\phi_m\alpha_+(0)a - i\lambda_3\alpha_-(0)a. \tag{A4}$$

We shall express the expectation value of the current by Green's functions $\underline{G}_{\nu\mu}(t)$ where the indices refer to the fermion local operators $\nu=(a,b)=(1,2)$ and $\mu=[\alpha_+(x=0),\alpha_-(x=0),\beta_+(x=0),\beta_-(x=0)]=(1,2,3,4)$. Using Eq. (A4) the current expectation value reads

$$\langle I(t=0)\rangle = -2\sqrt{T_K}\sin\phi_m G^<_{b\alpha_+}(t=0) - \lambda_2\cos\phi_m G^<_{a\alpha_+}(t=0) - \lambda_3 G^<_{a\alpha_-}(t=0). \tag{A5}$$

We construct the exact GFs [appearing in Eq. (A5)] from the free GFs calculated at $t=-\infty$ $(H_1=0)$: for $\mu,\mu'=(\alpha_+,\alpha_-,\beta_+,\beta_-)=(1,2,3,4)$, one finds

$$(G^{R,A})^{(0)}_{\mu\mu'} = \frac{\mp i}{2}\delta_{\mu\mu'},$$

$$(G^<)^{(0)}_{\alpha_\pm,\alpha_\pm} = i\tilde{f}(\omega), \quad (G^<)^{(0)}_{\beta_\pm,\beta_\pm} = if(\omega),$$

$$(G^<)^{(0)}_{\alpha_\pm,\alpha_\mp} = \pm\frac{1}{2}[f(\omega-eV)-f(\omega+eV)]. \tag{A6}$$

Here $f(x)=(1+e^{x/T})^{-1}$, $\tilde{f}(x)=\frac{1}{2}[f(x+eV)+f(x-eV)]$. Note that the voltage couples the two Majorana fermions α_+ and α_-, and here we assumed a bandwidth $\gg\omega,V,T$. The free GF for the local Majorana fermions $\nu=(a,b)=(1,2)$ is $(G^R)^{(0)}_{\nu\nu'}=\delta_{\nu\nu'}(\omega+i\delta)^{-1}$, where δ is a positive infinitesimal. We write H_1 in a convenient form

$$H_1 = -i\sqrt{T_K}\lambda_1 ab - i\sum_{\mu=1}^{4}\sum_{\nu=1}^{2}(\alpha_+,\alpha_-,\beta_+,\beta_-)_\mu\Lambda_{\mu\nu}(a,b)_\nu,$$

where

$$(\Lambda_{11},\Lambda_{21},\Lambda_{31},\Lambda_{41}) = (-\lambda_3,\lambda_2\cos\phi_m,0,-\lambda_2\sin\phi_m),$$

$$(\Lambda_{12},\Lambda_{22},\Lambda_{32},\Lambda_{42}) = 2\sqrt{T_K}(0,\sin\phi_m,0,\cos\phi_m). \quad (A7)$$

We obtain the full GF $\underline{G}_{\nu\nu'}$ for $\nu,\nu'=a,b$ as follows. First suppose $K=K_c$ ($\lambda_1=0$); we denote the different GFs and self-energies in this case as $\bar{G}$, $\bar{\Sigma}$, respectively. At $K=K_c$ the self-energy matrix is

$$\bar{\underline{\Sigma}}_{\nu\nu'} = -\Lambda_{\mu\nu}\underline{G}^{(0)}_{\mu\mu'}\Lambda_{\mu'\nu'} \quad \text{(repeated indices summed)}.$$

Equations (A6) and (A7) give $\bar{\Sigma}^R_{aa}=-\frac{i}{2}(\lambda_2^2+\lambda_3^2)$, $\bar{\Sigma}^R_{bb}=-2iT_K$, and $\bar{\Sigma}^R_{ab}=\Sigma^R_{ba}=0$. Equation (A2) gives $\bar{G}^R_{aa}=(\omega+i\frac{\lambda_2^2+\lambda_3^2}{2})^{-1}$, $\bar{G}^R_{bb}=(\omega+2iT_K)^{-1}$, and $\bar{G}^R_{ab}=\bar{G}^R_{ba}=0$. For energies $\ll T_K$ we can approximate $\bar{G}^R_{bb}=(2iT_K)^{-1}$. For the lesser GF Eq. (A3) gives

$$\bar{G}^< = \bar{G}^R\bar{\Sigma}^<\bar{G}^A, \quad (A8)$$

where matrix equation and multiplication in ab space are understood.

For $K\neq K_c$ the full matrix GF $\underline{G}_{\nu\nu'}$ can be calculated from the series Eq. (A1) where $G^{(0)}\to\bar{G}$ and

$$\Sigma^R = \Sigma^A = \sqrt{T_K}\lambda_1\tau^y,$$

$\Sigma^<=0$, where τ^y acts in ab space. Equation (A2) gives

$$G^R_{aa}(\omega) = (\omega + i\lambda^2/2)^{-1},$$

$$G^R_{bb}(\omega) = (2iT_K)^{-1} - (\lambda_1^2/4T_K)(\omega + i\lambda^2/2)^{-1},$$

$$G^R_{ab(ba)}(\omega) = \mp(\lambda/2\sqrt{T_K})(\omega + i\lambda^2/2)^{-1}.$$

For $G^<$, since $\Sigma^<=0$ we are left with the second term of Eq. (A3), which simplifies to [using Eqs. (A2) and (A8)]

$$G^< = G^R\bar{\Sigma}^<G^A,$$

where matrix equation and multiplication in ab space are understood.

The GFs appearing in the current Eq. (A5) satisfy the Dyson equation,

$$\underline{G}_{\nu\mu}(t=0) = \int\frac{d\omega}{2\pi}\sum_{\nu',\mu'}\underline{G}_{\nu\nu'}(\omega)i\Lambda_{\mu'\nu'}\underline{G}^{(0)}_{\mu'\mu}(\omega). \quad (A9)$$

To evaluate $\underline{G}^<_{\nu\mu}$ we use the identity $(\underline{A}\underline{B})^<=\underline{A}^<\underline{B}^A+\underline{A}^R\underline{B}^<$.

We encounter two types of integrals for the current,

$$I'[V,T,\lambda^2] = i\int d\omega\frac{f(\omega-eV)-f(\omega+eV)}{\omega+i\lambda^2/2} = \mathrm{Im}\,\psi\left(\frac{1}{2}+\frac{\lambda^2/2+ieV}{2\pi T}\right),$$

$$I''[V,T,\lambda^2] = \int d\omega\frac{f(\omega)-\tilde{f}(\omega)}{\omega+i\lambda^2/2} = \psi\left(\frac{1}{2}+\frac{\lambda^2/2}{2\pi T}\right) - \mathrm{Re}\,\psi\left(\frac{1}{2}+\frac{\lambda^2/2+ieV}{2\pi T}\right).$$

Note that $I'[V]=-I'[-V]$, and $I''[V]=I''[-V]$. From these results one can readily obtain the result for the nonlinear conductance [Eq. (6.1)].

[1] D. Goldhaber-Gordon, H. Shtrikman, D. Mahalu, David Abusch-Magder, U. Meirav, and M. A. Kastner, Nature (London) **391**, 156 (1998).
[2] S. M. Cronenwett, T. H. Oosterkamp, and L. P. Kouwenhoven, Science **281**, 540 (1998).
[3] M. Grobis, I. G. Rau, R. M. Potok, H. Shtrikman, and D. Goldhaber-Gordon, Phys. Rev. Lett. **100**, 246601 (2008).
[4] A. Schiller and S. Hershfield, Phys. Rev. B **58**, 14978 (1998).
[5] R. M. Konik, H. Saleur, and A. Ludwig, Phys. Rev. B **66**, 125304 (2002).
[6] P. Mehta and N. Andrei, Phys. Rev. Lett. **96**, 216802 (2006).
[7] E. Sela and I. Affleck, Phys. Rev. Lett. **102**, 047201 (2009).
[8] B. A. Jones, C. M. Varma, and J. W. Wilkins, Phys. Rev. Lett. **61**, 125 (1988).
[9] I. Affleck and A. W. W. Ludwig, Phys. Rev. Lett. **68**, 1046 (1992).
[10] I. Affleck, A. W. W. Ludwig, and B. A. Jones, Phys. Rev. B **52**, 9528 (1995).
[11] J. Gan, Phys. Rev. Lett. **74**, 2583 (1995); Phys. Rev. B **51**, 8287 (1995).
[12] T. Ivanov, Europhys. Lett. **40**, 183 (1997); T. Pohjola, J. König, M. M. Salomaa, J. Schmid, H. Schoeller, and Gerd Schön, *ibid.* **40**, 189 (1997).
[13] A. Georges and Y. Meir, Phys. Rev. Lett. **82**, 3508 (1999).
[14] W. Izumida and O. Sakai, Phys. Rev. B **62**, 10260 (2000).
[15] Ramón Aguado and David C. Langreth, Phys. Rev. Lett. **85**, 1946 (2000); R. López, R. Aguado, and G. Platero, *ibid.* **89**, 136802 (2002).
[16] M. Pustilnik and L. I. Glazman, Phys. Rev. Lett. **85**, 2993 (2000); T. Aono and M. Eto, Phys. Rev. B **63**, 125327 (2001).
[17] G. Zaránd, Chung-Hou Chung, P. Simon, and Matthias Vojta, Phys. Rev. Lett. **97**, 166802 (2006).
[18] R. M. Konik, Phys. Rev. Lett. **99**, 076602 (2007).
[19] C.-H. Chung and W. Hofstetter, Phys. Rev. B **76**, 045329 (2007).
[20] B. A. Jones, B. G. Kotliar, and A. J. Millis, Phys. Rev. B **39**, 3415 (1989).
[21] O. Sakai and Y. Shimizu, J. Phys. Soc. Jpn. **61**, 2333 (1992); **61**, 2348 (1992).
[22] R. Landauer, IBM J. Res. Dev. **1**, 223 (1957).
[23] See, for example, I. Affleck and A. W. W. Ludwig, Nucl. Phys. B **360**, 641 (1991).
[24] J. L. Cardy, Nucl. Phys. B **324**, 581 (1989).
[25] J. L. Cardy and D. C. Lewellen, Phys. Lett. B **259**, 274 (1991).
[26] I. Affleck and A. W. W. Ludwig, Phys. Rev. B **48**, 7297 (1993).
[27] S. Ghoshal and A. Zamolodchikov, Int. J. Mod. Phys. A **9**, 3841 (1994); **9**, E4353 (1994).
[28] R. Konik, A. LeClair, and G. Mussardo, Int. J. Mod. Phys. A **11**,

2765 (1996).
[29] A. LeClair, F. Lesage, S. Sachdev, and H. Saleur, Nucl. Phys. B **482**, 579 (1996).
[30] J. M. Maldacena and A. W. W. Ludwig, Nucl. Phys. B **506**, 565 (1997).
[31] G. Zaránd and J. von Delft, Phys. Rev. B **61**, 6918 (2000).
[32] M. Pustilnik, L. Borda, L. I. Glazman, and J. von Delft, Phys. Rev. B **69**, 115316 (2004).
[33] H. Jeong, A. M. Chang, and M. R. Melloch, Science **293**, 2221 (2001).
[34] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)].